

# Toward a theory of curvature-scaling gravity

Hoang Ky Nguyen\*

New York City, USA

(February 26, 2013)

## Abstract

A salient feature of Hořava-Lifshitz gravity is the anisotropic time variable [1]. We propose an alternative construction of the spacetime manifold which naturally enables anisotropy in time scaling. Our approach promotes the role of curvature: the Ricci scalar  $\mathcal{R}$  at a given point on the manifold sets the length scales for physical processes – including gravity – in the local inertial frames enclosing that point. The manifold is viewed as a patchwork of local regions; each region is Lorentz-invariant and adopts a variable local scale which we call the Ricci length  $a_{\mathcal{R}}$  defined as  $a_{\mathcal{R}} \triangleq |\mathcal{R}|^{-1/2}$ . In each local patch, the length scales of physical processes are measured relatively to the local Ricci length  $a_{\mathcal{R}}$ , and only their dimensionless ratios participate in the dynamics of the physical processes. The time anisotropy arises from the requirement that the form – but not necessarily the parameters – of physical laws be unchanged under variations of the local Ricci length as one moves along any path on the manifold. The time duration is found to scale as  $dt \propto a_{\mathcal{R}}^{3/2}$  whereas the spatial differential scales as  $dx \propto a_{\mathcal{R}}$ .

We show how to conjoin the local patches of the manifold in a way which respects causality and other requirements of special relativity, as well as the equivalence principle and the general covariance principle. In our approach, all of Einstein's insights are preserved but the parameters of physical laws are only valid locally and become functions of the prevailing Ricci scalar. As such, the parameters are allowed to vary on the manifold, together with the Ricci scalar. This alternative construction of the manifold permits a unique choice for the Lagrangian of gravity coupled with matter. Curvature thus acquires a new privileged status – it is actively involved in the dynamics of physical processes by setting the scale for them; hence the name curvature-scaling gravity.

In vacuo curvature-scaling gravity takes the form of quadratic Lagrangian  $\mathcal{R}^2$ , which adopts a larger set of solutions superseding all solutions to the field equations based on the Einstein-Hilbert action  $\mathcal{R}$ . We provide two static spherically symmetric solutions: one solution connects our theory to Mannheim-Kazanas's conformal gravity-based solution [2] which Mannheim argues to account for the galactic rotation curves [3–6]; the other solution leads to novel properties for Schwarzschild-type black holes. Additionally, we apply curvature-scaling gravity to address an array of problems encountered in cosmology, and discuss its implications in the quantization of gravity.

## Contents

1	Postulate on the dependence of local length scales on the scalar curvature of spacetime . . . . .	2
2	Equation of motion of a point mass in a given curvature-scaling gravitational field . . . . .	7
3	Lagrangian and the field equations of curvature-scaling gravity in vacuo . . . . .	9
4	Connection of curvature-scaling gravity to Mannheim's conformal gravity-based treatment of galactic rotation curves . . . . .	12
5	Logical inferences from the postulate of Ricci scalar as dynamical scale-setter . . . . .	16
5.1	The anisotropy in time scaling . . . . .	16
5.2	The first conceptual departure . . . . .	17
5.3	The second conceptual departure and the preservation of causality . . . . .	18
5.4	Applications of the anisotropic time scaling in cosmology: An invitation . . . . .	20
6	A nontrivial solution to the curvature-scaling field equation and its consequences in the physics of black holes . . . . .	24
7	Coupling of curvature-scaling gravity with matter: a departure from conventional construction of Lagrangian . . . . .	27
8	Implications of curvature-scaling gravity in cosmography . . . . .	31
8.1	The modified Robert-Walker metric . . . . .	31
8.2	Modification to Lemaitre's redshift formula . . . . .	32
8.2.1	For the traditional RW metric: . . . . .	32
8.2.2	For the modified RW metric: . . . . .	32
8.3	The modified Hubble law and the corrected value for Hubble constant . . . . .	33
8.4	The modified distance-redshift relationship . . . . .	34
8.4.1	For the Lambda-CDM model: . . . . .	34
8.4.2	For curvature-scaling gravity: . . . . .	35
8.5	The modified luminosity-redshift relationship . . . . .	35

\*Electronic address: HoangNguyen7@hotmail.com

8.5.1	For the Lambda-CDM model: . . . . .	35
8.5.2	For curvature-scaling gravity: . . . . .	36
8.6	A critical analysis of Type Ia supernovae data . . . . .	37
8.7	An alternative interpretation to Type Ia supernovae data based on curvature-scaling gravity . . . . .	38
9	Implications of curvature-scaling gravity in cosmology . . . . .	40
9.1	The evolution of the cosmic scale . . . . .	40
9.2	Resolution to the age problem . . . . .	41
9.3	Resolution to the horizon problem . . . . .	42
9.4	Resolution to the flatness problem . . . . .	42
9.5	Resolution to Dicke’s “runaway density parameter” problem or the oldness problem . . . . .	43
9.6	On the shortcomings of the Friedmann model and its related supplementaries . . . . .	43
10	Conclusions and outlook of curvature-scaling gravity in quantum gravity . . . . .	46
A	Derivation of the scaling rule for time duration . . . . .	50
B	Buchdahl’s treatment of $\mathcal{R}^2$ gravity revisited . . . . .	51
C	An explicit solution for spherically symmetric case . . . . .	55
D	A perturbative solution to the vacuo field equation: The anomalous curvature . . . . .	59
E	The Ricci scalar in the Robertson-Walker metric and the modified Robertson-Walker metric . . . . .	63
F	Three forms of the time duration paradox . . . . .	64
G	On the redshift of photons from distant galaxies . . . . .	65

## 1 Postulate on the dependence of local length scales on the scalar curvature of spacetime

Einstein’s vision of spacetime consists of a manifold with (pseudo-)Riemannian geometry. The curvature of the manifold is determined by the distribution of matter and is governed by the Einstein equations of the metric components  $g_{\mu\nu}$ . At a given point on the manifold, the metric can be temporarily and locally approximated by a Minkowskian one which in effect eliminates the effect of curvature at the point. In the local inertial coordinate system, test particles follow geodesics and all physical laws of non-gravitational origin obey the Lorentz symmetry as established in special relativity. Gravity is interpreted as the effects of curvature acting non-locally on the geodesics; in the local inertial frame the effects of gravity disappear. These aspects of spacetime and gravity have been theoretically established in general relativity and experimentally verified with great precision for solar system. The post-Newtonian phenomenology of gravity includes the three classic tests – the precession of Mercury’s perihelion, the gravitational redshift, the bending of starlight across the Sun disc – and later added to the list of successes for Einstein’s theory is the decaying orbits of binary pulsars.

In this report, we aim to extend Einstein’s insights further. At the heart of our undertaking is the role of curvature which we theorize to be more profound than so far being perceived in literature and carried out in practice. We are guided by the following observation.

### Our observation:

The equivalence principle states that at a given point on the manifold one can choose a coordinate system (a “tangent” frame) such that the effects of gravity are instantaneously and locally eliminated. Einstein was inspired by the example of a free falling elevator to arrive at this insightful conclusion. The equivalence of acceleration and local gravity establishes the equality between the inertial mass and the passive gravitational mass of an object. Non-gravitational forces act as if they were in a flat spacetime, with physical phenomena within the local inertial frame obeying special relativity together with other laws with non-gravitational origin, such as quantum mechanics. What puzzles us, however, is that whilst on the one hand, the presence of a real gravitation field – i.e., the curved spacetime – in the region of the elevator cannot be eliminated since curvature is characterized by the 20 degrees of freedom inherent in the Riemann tensor (in particular, the Ricci scalar being an invariant and thus manifest in all coordinate systems), on the other hand, curvature curiously plays no direct role whatsoever in the dynamics of the non-gravitational physics in the local region once it is moved to an inertial frame. The non-gravitational processes and laws in any of these “tangent” frames are oblivious to the value of the Ricci scalar at that point. Whilst this is a proceeding customarily employed in general relativity and, de facto, all gravitational theories that followed since, we suggest that the curvature plays a greater role than traditionally assumed: it explicitly sets the size of physical objects in the “tangent” frame – a role that has been overlooked in conventional theories of gravity. That is to say, being of the unit  $[\text{length}]^{-2}$ , the Ricci scalar should determine the length scales of physical processes that take place in the tangent region.

Let us define for each given point on the manifold a new length – and name it the Ricci length  $a_{\mathcal{R}}$  – taken to be the inverse of the square root of the absolute value of the Ricci scalar at the point:  $a_{\mathcal{R}} \triangleq |\mathcal{R}|^{-1/2}$ . The Ricci length is itself an invariant, i.e., having the same value for all choices of the coordinate system, and is obviously of dimension of length. We shall assert that the Ricci length is of a more fundamental status than all other length scales, and that all other length scales are denominated in terms of the Ricci length. That is to say, the length scale of any physical process in a local

tangent region is dynamically determined by the Ricci scalar which in turn can vary from one point to the next on the manifold (with the dynamics of  $\mathcal{R}$  being related to that of  $g_{\mu\nu}$ ), and only the dimensionless relative ratio between the length scale of the process and the Ricci length is relevant for the dynamics of the process. (We stress that our ideas are not related to the dilaton or Brans-Dicke theories. First, the scale setter in our approach is the Ricci scalar itself, instead of an auxiliary field with its own dynamics. Second, as will be shown in Section 5, our approach enables anisotropy in time scaling – an important feature absent in dilaton and Brans-Dicke.)

More concretely, with a given metric  $g_{\mu\nu}$  of the signature convention  $(+, -, -, -)$ , in each local patch, a test particle of mass  $m$  travels on a timelike geodesics ( $ds^2 \triangleq g_{\mu\nu}dx^\mu dx^\nu > 0$ ):

$$\frac{d^2x^\mu}{ds^2} + \Gamma_{\nu\lambda}^\mu \frac{dx^\nu}{ds} \frac{dx^\lambda}{ds} = 0, \quad \mathcal{R}_{\mu\nu\lambda\sigma} \neq 0 \quad (1)$$

for an arbitrary choice of coordinate  $x^\mu$ . The equation of motion (1) is derivable from the action for the test particle used in the standard theory of relativity

$$\mathcal{S} \simeq \int ds. \quad (2)$$

This action is built from an invariant, the proper distance  $ds$ , thus yielding the equation of motion (1) in the tensorial form, as required by the general covariance principle. At a given point  $P$ , if one chooses a coordinate such that the Christoffel  $\Gamma_{\nu\lambda}^\mu$ 's vanish (i.e., the inertial frame), the effects of curvature – interpreted as the effects of gravity – are locally eliminated around  $P$  since the trajectory of the test particle is a geodesics, and all physical phenomena of non-gravitational origin look exactly like ones in the Minkowskian metric, i.e., satisfying all the elements of special relativity. This is the essence of the equivalence principle. Yet, as we noted before, the curvature curiously drops out of the geodesics equation (1) when  $\Gamma_{\nu\lambda}^\mu = 0$  and is decidedly absent in action (2). This is not necessarily so, however: we can fully retain the validity of the equivalence principle even if the proper distance in action (2) is divided by the Ricci length,  $a_{\mathcal{R}}$  – itself an invariant; namely, the integrand is replaced by  $ds/a_{\mathcal{R}}$  – a combination of invariants. That is to say:

$$\mathcal{S} \simeq \int \frac{ds}{a_{\mathcal{R}}}. \quad (3)$$

The equivalence principle in no way prevents this possibility; the extra divisor  $a_{\mathcal{R}}$  does not imperil the equivalence principle which is local in nature<sup>1</sup>. Moreover, this new choice for the integrand,  $ds/a_{\mathcal{R}}$ , fully respects all the requirements of special relativity in the local inertial frame. In particular, thanks to the local Lorentz symmetry, the order of precedence of causally-connected events is an invariant; as such, causality is protected globally. In addition, with such an integrand being built from invariants, the proper distance and the Ricci scalar, the general covariance principle is automatically and globally ensured. As such, all of Einstein's insights of relativity are respected.

We therefore conclude that the de facto adoption of  $ds$  in action (2) is not the only choice allowable, with  $ds/a_{\mathcal{R}}$  as in action (3) being the other acceptable option as well. We project that conventional wisdom has been under-appreciating the role of curvature, and that the theory of general relativity can be made tighter: not only does the scalar curvature determine the action (such as action (3) above, which equals  $\int ds |\mathcal{R}|^{1/2}$ , and the Einstein-Hilbert action  $\int d^4x \sqrt{-\det g} \mathcal{R}$ ), the curvature can further partake in the dynamics of the physical phenomena and processes in the local inertial frame by setting the scale for them. We thus upgrade the curvature to an eminent role, a role which we believe has been overlooked in the development of gravitational theories: the Ricci scalar directly sets the length scales for all physical phenomena that take place in the local inertial frame (and indeed all non-inertial frames since the Ricci scalar is an invariant.) In other words, all length scales<sup>2</sup> of physical processes in a local pocket of spacetime are pegged to the Ricci length prevailing in the pocket. As intuitive as our arguments are, we nonetheless cast them in two following postulates.

### Postulate I: *On the invariance of the form of physical laws*

*Physical laws retain their forms – but not necessarily their parameters – in every local region on the spacetime manifold.*

### Postulate II: *On the role of the Ricci scalar as a local scale-setter*

*The Ricci length, defined via the Ricci scalar  $\mathcal{R}$  as  $a_{\mathcal{R}} \triangleq |\mathcal{R}|^{-1/2}$ , is the common denominator for all length scales of physical processes – gravitational and non-gravitational alike – that take place in a local region. Physical laws are dependent only on the dimensionless ratios of lengths normalized by the Ricci length.*

<sup>1</sup> The preservation of the equivalence principle in this manner was previously recognized by Mannheim in his review [3]. See Eq. (11) and the comment before Eq. (145) of [3] in which the test particle is coupled with a scalar field  $S$ :  $\mathcal{S} \simeq \int ds S(x)$ ; this action formally does not imperil the equivalence principle. In our approach, we furnish the physical meaning for the scalar field in which  $S$  is set to be inverse of the Ricci length.

<sup>2</sup> Including the Bohr radius which is governed by quantum mechanics.

Postulate (I), at the first look, is nothing but a reinforcement of special relativity and the equivalence principle. In each local region, all physical processes satisfy special relativity (i.e., the Lorentz symmetry) as well as all other established laws in the quantum realm. For example, the Dirac equation must retain its form for every local pocket. Our emphasis, however, is the allowance for the parameters of the physical laws to vary from one local region to the next on the manifold. Indeed, since the equivalence principle is local in nature, special relativity is only required to hold locally. What we further enforce here is that the parameters of physical laws be valid locally too. What is left unaddressed in Postulate (I) is how the parameters are specified for each local region.

Postulate (II) quantifies how the parameters are determined in each local pocket. To do this, it requests that the length scale of every physical process be pegged into the Ricci length in the local pocket. If the Ricci length varies, the length scale of any physical process must vary in exact proportion. As a result, physical laws must be expressed in terms of dimensionless ratios of lengths denominated by the Ricci length. Whereas the form of the physical law that governs a process is unchanged – per Postulate (I) – as we move from one pocket to another on the manifold, the length scale intrinsic in the law must not; rather, it has to adapt to the prevailing value of the Ricci length, per Postulate (II). The Ricci length – holding a higher status than all other length scales – acquires a dynamics from the evolution of the spacetime manifold, viz. the gravitational field equations, which are to be derived in this report. As we shall see, the power of these two postulates goes beyond non-gravitational laws; they too embrace gravitational laws by prescribing the dynamical equations for  $g_{\mu\nu}$  as well, which is the ultimate objective of this report.

We thus arrive at a new depiction of the spacetime manifold: the spacetime manifold constitutes a patchwork of local regions, each of which strictly obeys special relativity and established non-gravitational physics (be it the standard quantum field theory with  $SU(3) \times SU(2) \times U(1)$  gauge symmetries plus spontaneous symmetry breaking, the phonons and quantum Hall effect in solids, or the tunneling effect in nuclear decays.) There is no fundamental length scale applicable globally to the whole manifold, but each individual region adopts a local length scale  $a_{\mathcal{R}}$  set by the Ricci scalar in the region. All regions are indistinguishable, however; the observer cannot tell which region he lives in by using only the measurement apparatus that are available within his surroundings. It is permissible, as we shall show in this report, to conjoin the local patches in a way which respects (global) causality and the general covariance principle. Our new construction of the spacetime manifold is a departure from Einstein's theory while we preserve all of his insights – the Lorentz symmetry, the Michelson-Morley finding, the relativity principle, the equivalence principle, and the general covariance principle – in our construction. It is the relinquishment of a global length scale – but instead adopting a dynamical length scale associated with the Ricci scalar – that is the only new crucial element in our approach.

In plain language, let us reconsider Einstein's gedanken free falling elevator. Let an observer residing within the elevator throw a brick. He will observe the brick moving on a straight line at constant speed, using his own ruler and clock; i.e., the pulling effects of the Earth under his feet is absent. This is the essence of the equivalence principle. However, we go one step further. We assert that the scalar curvature further determines the size of each object in the elevator (including the brick and the observer's ruler) and the oscillatory rate of the observer's wristwatch.<sup>3</sup> In so doing, we strengthen the position of the curvature within Einstein's theory by promoting the Ricci scalar to a prominent status: not only does it play a crucial part in the underlying geometry, it actively participates in the dynamical process of physical phenomena by setting the scale for them. This new element – encapsulated in Postulate (II) – is the centerpiece for all of our subsequent considerations. This report is devoted to examine the ramifications of this idea.

### The strategy:

Our ultimate objective is to find the field equations for gravity. Due to Postulate (II), the field equations will necessarily be different from those in Einstein's gravity. To give the reader a sense of what shall be expanded in this report, below is the outline of our program. Only dimensionless combinations of lengths denominated by the Ricci length, such as  $ds/a_{\mathcal{R}}$  or  $dx^\mu/a_{\mathcal{R}}$ , shall enter the action. Beside the conventional way that the metric components enter the action (via the minimal coupling procedure which allows  $g_{\mu\nu}$  to enter the action via the Jacobian  $\sqrt{-\det g}$ , the covariant derivatives  $\nabla_\mu$ , and the contrarian derivatives  $\nabla^\mu \triangleq g^{\mu\nu}\nabla_\nu$ ), the metric will also accompany the Ricci length to enter the action via two additional entrances:

- Via the Lagrangian of matter fields which is comprised of derivatives of the matter fields. The (covariant) derivatives  $\nabla_\mu$  will be normalized by the Ricci length; i.e., they are replaced as  $\nabla_\mu \rightarrow a_{\mathcal{R}}\nabla_\mu = |\mathcal{R}|^{-1/2}\nabla_\mu$ ;
- Via the 4-volume  $d^4x$  in the volume element  $d^4x\sqrt{-\det g}$ , the first part of which is replaced as  $d^4x \rightarrow (dx/a_{\mathcal{R}})^4 = d^4x\mathcal{R}^2$ .

All these routes combined open the door for matter fields to couple with gravity in an organic, natural, and unique way. Our procedure outlined above is an important breakaway from the standard procedure of minimal coupling, viz.  $\int d^4x\sqrt{-\det g}\mathcal{L}_m$ .

<sup>3</sup> This latter point holds the key to the time anisotropy alluded to in the Abstract, to be discussed in Section 5.

Mathematically, the global structure of the manifold remains to be torsion-free pseudo-Riemannian geometry, with its curvature – the Riemann tensor  $\mathcal{R}^\mu_{\nu\lambda\sigma}$  and its contracted derivatives, the Ricci tensor  $\mathcal{R}_{\mu\nu}$  and Ricci scalar  $\mathcal{R}$  – obey the well-developed mathematics of Riemannian geometry and diffeomorphism.

Conceptually speaking, unlike the equivalence principle which was inspired by Einstein’s gedanken elevator, the conventional choice of  $ds$  in action (2) did not appeal to any fundamental principle, but rather has been a convenient choice <sup>4</sup>. By adopting the alternative  $ds/a_{\mathcal{R}}$  as in action (3) we resort to a requisition that local scale no longer be an omnipresent prefixed property but instead be dynamically determined by the very structure of spacetime itself, a requisition which lends ontological and aesthetical appeals to the theory.

### The criteria:

Our approach thus is an alternative construction of the spacetime manifold which we shall explicitly build. The success or failure of such a permissible construction needs be judged against the following list of criteria:

1. Will it embrace the well-established principles, the most important of which are the relativity principle, the Lorentz invariance (i.e., the Michelson-Morley experiment), the equivalence principle, and the general covariance principle? Also being included in the list are the laws of quantum physics (quantum mechanics and quantum field theories of the electroweak and strong forces – namely, the laws with non-gravitational origin). <sup>5</sup>
2. Will it protect causality? <sup>6</sup>
3. Will it give rise to a form of gravity which guarantees to recover the precise solar system tests as well as the decaying orbits of binary pulsars?
4. Will it offer new interesting physics together with testable predictions? Will it help – in the Occam’s Razor sense – solve problems and difficulties that plague cosmology, astrophysics, quantum gravity?
5. Will it offer new conceptual insights?
6. Last but not least, at what costs, mathematics-wise?
7. And concept-wise?

The answers to the first five criteria are affirmative. Our approach imposes no structural damage to the existing physical laws. Every local pocket of spacetime, by construction, satisfies special relativity (that is, including the Michelson-Morley experiment, the constancy of light speed regardless of the state of the observer and/or the light emitter) and satisfies the quantum laws. The laws retain exactly the same form – but not necessarily the parameters – in all local pockets. There is no local pocket that is more privileged than any others. The parameters of the laws, being dynamically pegged to the Ricci scalar, however, can inherit the dynamics of the Ricci scalar, which itself is governed by the field equations of the metric  $g_{\mu\nu}$ . But their variabilities in no way imperil the well-tested principles of relativity. With the actions being constructed from invariants, the general covariance principle is ensured as well.

Causality is protected since in every local region the Lorentz symmetry remains intact. No coordinate transformation can frustrate the order of precedence of any pair of causally-related events in the local region, and thus causality globally holds for any pair of time-like events on the manifold. Time-like trajectories and space-like trajectories do not mix. Null geodesics remain null-geodesics, separating the set of time-like trajectories from the set of space-like trajectories.

We shall show that the new Lagrangian of gravity duly recovers post-Newtonian phenomenology. This is because in vacuo the new gravity to be built in this report resemblances the quadratic Lagrangian  $\mathcal{R}^2$  which is known to accept all solutions to the Einstein field equations (based on the Einstein-Hilbert action  $\mathcal{R}$ ) as its subset. As long as the Ricci scalar, and thus the Ricci length, varies little in the solar system, the effects of the Ricci length  $a_{\mathcal{R}}$  on the three classic tests of Einstein’s gravity as well as the decaying orbits of binary pulsars should be negligible.

There will be new interesting outcomes which we shall cover in this report. These new outcomes will have significant impacts on a range of fundamental issues encountered in astrophysics, cosmography, cosmology, and the quantization of gravity.

<sup>4</sup> The Einstein-Hilbert action, being a second-order theory, has a well-posed initial value problem for the metric, in which “coordinates” and “momenta” specified at an initial time can be used to predict future evolution. This is often cited as an advantage, if not the major rationale for the theory. A gravitational theory which involves higher-derivative terms would require not only those data, but also some number of derivatives of the momenta, thus becoming intractable both conceptually and practically. We shall show in Section 3 that this is not the case for curvature-scaling gravity which also only requires “coordinates” and “momenta” as its Cauchy data.

<sup>5</sup> The laws of gravity are temporarily absent in this list precisely because we are embarking on a search for the laws for gravity, away from Einstein’s gravity. Yet we still want to retain the equivalence principle and the general covariance principle in the list of well-tested and all-embracing principles to be satisfied.

<sup>6</sup> We single out the causality principle as this is a very touchy issue which merits special attention.

Mathematically, the costs in formalism are virtually nil. We can fully enjoy the machinery of Riemannian geometry and diffeomorphism developed in the last 150 years or so. This is because the underlying geometry remains to be torsion-free pseudo-Riemannian (not Weyl, for example).

Regarding Point (7), we must however note that there are conceptual departures which we shall elaborate in due course. The conceptual route we are undertaking logically leads to a number of substantial adjustments in one's view regarding the underpinnings of physical laws.

Before we embark on the program, we have three further remarks:

- Our approach is not related to the dilaton or Brans-Dicke theories. The scale setter is the Ricci scalar the dynamics of which is that of the metrics  $g_{\mu\nu}$ , instead of an auxiliary scalar field that lives on the manifold and has its own dynamics. Further fundamental differences between our approach and Brans-Dicke/dilaton, such as the time anisotropy which arises from our approach and is absent in Brans-Dicke/dilaton, will be elucidated in Section 5.
- Comparison with conformal gravity: Our approach should not be mistaken as any equivalent form as the well-studied conformal gravity. Indeed, our approach is precisely the opposite: whereas in conformal gravity, the curvature of spacetime is “gauged” away by conformal transformations, our approach upgrades the Ricci scalar to an eminent role: it sets the length scales for all physical phenomena including gravity from one spacetime point to the next.
- An objection would be that in vacuo spacetime is Ricci-flat (i.e.,  $\mathcal{R} = 0$ ) according to the Einstein field equations, thereby rendering the Ricci length infinite. However, this is a premature objection for two reasons: (i) The field equations for curvature-scaling gravity in vacuo generally admits spacetime configurations that have non-zero Ricci scalar. This is because the field equations in vacuo, as we shall see, echo the field equations of the quadratic Lagrangian  $\mathcal{R}^2$  which is known to admit a richer set of solutions than Einstein's gravity (based on  $\mathcal{R}$ ) does. (ii) The Robertson-Walker metric, considered to be applicable to the cosmos, generally has non-zero Ricci scalar. In the Robertson-Walker metric, even if space is flat, spacetime is not. Both of these reasons would confine the domain with vanishing Ricci scalar to a set of zero-measure.

Our report is structured as follows. Section 2 presents the Lagrangian and equation of motion for a test particle in curvature-scaling gravitational field; we also show how causality is protected in curvature-scaling gravity. Section 3 builds the Lagrangian of the curvature-scaling gravitational field in vacuo. Section 4 shows our solution to the curvature-scaling field equation in vacuo and its connection with Mannheim's phenomenological theory of galactic rotation curves [2–6]. Section 5 presents a series of logical consequences of our two postulates declared on page 3; in particular, we show how the anisotropic time scaling would arise from the two postulates by pure deduction. Section 6 presents a nontrivial solution of our curvature-scaling field equations and, combined with the logical deductions in Section 5, makes a prediction of a new type of black holes with novel properties. Section 7 builds the Lagrangian of curvature-scaling gravity coupled with matter. Section 8 details the implications of curvature-scaling gravity in cosmography; we address the (over)estimation problem with the Hubble constant (and related to which, the age problem), provide a critical reassessment of Type Ia supernovae data, and work out an alternative interpretation of these data bypassing the conventional explanation based on accelerated expansion. Section 9 explores the consequence of curvature-scaling gravity in cosmology; we show how it helps resolve the cosmological constant problem, the horizon problem, the flatness problem, among others while circumventing the inflationary expansion scenario. Section 10 summarizes the theory and offers an outlook how curvature-scaling gravity could offer a reasonable starting point toward quantum gravity. All the technical details are presented in the Appendices.

## 2 Equation of motion of a point mass in a given curvature-scaling gravitational field

Consider a test point mass  $m$  moving in a given metric  $g_{\mu\nu}$ . In the standard theory of general relativity, its action is well-known

$$\mathcal{S} = -mc \int ds \quad (4)$$

in which  $c$  is the speed of light and the length element  $ds^2 = g_{\mu\nu}dx^\mu dx^\nu > 0$ . We prefer to rewrite it as

$$\mathcal{S} \simeq \int ds \quad (5)$$

since only the integrand will be of relevance for our subsequent reasoning. The quantity of interest  $ds$  is of dimension of length. The particle is expected to move on a geodesics per the equation of motion (1). In curvature-scaling gravity, however, only the dimensionless ratio  $d\tilde{s} \triangleq ds/a_{\mathcal{R}}$ , in which  $a_{\mathcal{R}} = |\mathcal{R}|^{-1/2}$  is the local Ricci length, is the meaningful quantity to be used in the action. Therefore, we make the following replacement in action (5)

$$ds \rightarrow d\tilde{s} = \frac{ds}{a_{\mathcal{R}}} = |\mathcal{R}|^{1/2} ds. \quad (6)$$

The action now reads

$$\mathcal{S} \simeq \int d\tilde{s} = \int ds |\mathcal{R}|^{1/2}. \quad (7)$$

Upon a functional variation of  $x^\mu$  while holding  $g_{\mu\nu}$  fixed, we obtain the equation of motion for the test point mass:

$$\frac{d^2 x^\mu}{ds^2} + \Gamma_{\nu\lambda}^\mu \frac{dx^\nu}{ds} \frac{dx^\lambda}{ds} + \frac{1}{2\mathcal{R}} \frac{\partial \mathcal{R}}{\partial x^\nu} \left( g^{\mu\nu} + \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right) = 0. \quad (8)$$

Formally, this type of action has been considered in a more general form of action  $\int ds f(\mathcal{R})$ . So at the first look, action (7) and the equation of motion (8) offer nothing new, and the fact that the last term in Eq. (8) renders the motion non-geodesics has been well-understood. This perception is deceptive, though. The non-geodesics nature of the motion is the least interesting aspect in our approach. Our action (7) and Eq. (8) carry a very different physics from the formal  $\int ds f(\mathcal{R})$  considerations: action (7) arises from the change in local scale reflected in  $|\mathcal{R}|^{1/2}$  and Eq. (8) describes the motion of a point mass which adapts its scale dynamically as it moves on the manifold. At each new point where it arrives, the point mass “sees” its surrounding region at a new scale – the prevailing Ricci length  $a_{\mathcal{R}}$ . The physics around the point mass still obeys exactly special relativity as well as all other laws of non-gravitational origin (such as electromagnetism and quantum mechanics). However, the length scale for the physics in its surroundings is now a variable denominated by the prevailing Ricci length. All lengths are pegged to the Ricci length, and only their ratios with respect to the Ricci length appear in the action. Locally, the observer riding along with the point mass will not detect anything amiss; he will not be able to tell any difference in his surroundings since all physical laws retain their forms and all objects around him (including his ruler) adapt their scales accordingly to the Ricci length. For example, let the observer carry along a Michelson-Morley interferometer; at any point along his path, although the arms of the interferometer resize accordingly to the Ricci length, he will unambiguously replicate Michelson-Morley’s null result regarding the interference pattern in the interferometer. Therefore, the observer will not be able to tell which region he currently resides in. Globally, however, if he manages to exchange light signals with another observer residing at a location with a different value of the Ricci scalar, he would be able to detect the effects of the relative scale difference between the two locations. This is the new feature in our action (7) and equation of motion (8) <sup>7</sup>.

At any given point on its trajectory, the point mass and the observer will not feel any effect of the curvature (apart from the non-local tidal effect). Therefore, the equivalence principle, Lorentz invariance, and the relativity principle are satisfied locally. Note that Lorentz invariance and the relativity principle are only satisfied locally since the equivalence principle itself limits their validity to each individual local region as Einstein’s insight from his gedanken elevator requires. Since action (7) is built from invariants, the equation of motion (8) is explicitly tensorial and the general covariance principle is satisfied globally.

Causality deserves an examination on its own right, however. Note that due to the new aspect of length scale adaptation, which is missing in formal considerations (of  $\int ds f(\mathcal{R})$ ), we must provide a thorough look at causality as we shall do below.

<sup>7</sup> This feature absolutely does not mean that Gulliver and Lilliputians can cohabitate. They cannot. If Gulliver and a Lilliputian ever meet, each of them will adjust to the corresponding scale  $a_{\mathcal{R}}$  at their meeting location, thus both creatures having the same size.

### Causality:

For a time-like path, the total integral  $\int ds |\mathcal{R}|^{1/2}$  is an invariant since both terms  $ds$  and  $\mathcal{R}$  are invariants. If two events are causally connected, the order of the events is strictly an invariant regardless of the coordinate choice. Although the infinitesimal  $ds$  is “scaled” by a variable factor  $|\mathcal{R}|^{1/2}$ , in each local region, there cannot be any coordinate transformation that can flip the temporal order of events on a timelike trajectory.

Consider a timelike trajectory of a massive object. Along the path, consider a series of events  $A \equiv A_1, A_2, A_3, \dots, A_n \equiv B$  each being labeled by the time coordinate  $t_A \equiv t_1, t_2, t_3, \dots, t_n \equiv t_B$  in which event  $A_1$  precedes event  $A_2$  and so on. The order of time precedence dictates that

$$\begin{aligned} t_1 &< t_2 \\ t_2 &< t_3 \\ &\dots \\ t_{n-1} &< t_n \end{aligned} \tag{9}$$

At each time point  $t_k$  along the trajectory, the tangent frame is Lorentz-invariant. Therefore, no choice of the local coordinate at  $t_k$  can alter the time order in the pair  $\{t_k, t_{k+1}\}$ . Even though the differences  $t_2 - t_1, t_3 - t_2, \dots, t_n - t_{n-1}$  are dependent on the coordinate choice, the signs of them are not. The order of precedence for the series of events is strictly invariant regardless of the coordinate system. Moreover, since no coordinate transform can frustrate the order of precedence

$$t_A = t_1 < t_2 < t_3 < \dots < t_{n-1} < t_n = t_B, \tag{10}$$

causality holds globally for any pair of events,  $A$  and  $B$ , which are arbitrarily far apart on the timelike trajectory

$$t_A < t_B.$$

Causality is protected since the timelike trajectories ( $ds^2 > 0$ ) and the spacelike trajectories ( $ds^2 < 0$ ) form two disjoint sets. Null geodesics connecting two points  $C$  and  $D$  on the “lightcone”, for which the total  $\int_C^D ds = 0$  remains vanishing for all coordinate choices, are separatrix, strictly disconnecting the set of timelike trajectories from the set of spacelike trajectories.

There is another way to see why causality is protected. In each local region, since the Lorentz symmetry holds, the light speed is the upper limit of speed for all objects in the region. No massive particle can overcome the light speed barrier to jeopardize causality. Superluminality is strictly forbidden and causality is strictly enforced.

Note that the proof of causality relies solely on the requirement that each individual local frame be Lorentz-invariant. It does not make any resort to whether or not the intrinsic scale (in this case, the Ricci length) is fixed or is allowed to vary from one location to the next on the manifold. Also, it is very important to note that the proof of causality presented above does not require the light speed to be universal in all local regions. The conclusion regarding the preservation of causality even in the adaptation of length scales to the local Ricci length has far-reaching consequences, which we shall elaborate in Section 5.

### Effects on post-Newtonian phenomenology:

If the Ricci scalar  $\mathcal{R}$  varies on the manifold, the motion of a test point mass is globally non-geodesic, although the motion is locally geodesic. If the Ricci scalar varies little within a region under consideration, the last term in (8) should have minor effects. Since we do not expect the curvature to vary much within the solar system, corrections to solar system phenomenology are negligible, ensuring the recovery of post-Newtonian phenomenology. The task remains to show that the metric in solar system is almost the same as the Schwarzschild metric [7]:

$$ds^2 = \left(1 - \frac{r_s}{r}\right) (c dt)^2 - \frac{dr^2}{1 - \frac{r_s}{r}} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{11}$$

with  $r_s = \frac{GM}{2c^2}$  being the Schwarzschild radius represented in terms of the Newton gravitational constant  $G$ , the Sun mass  $M$  and the speed of light  $c$ . This task will be addressed in Section 4.



### 3 Lagrangian and the field equations of curvature-scaling gravity in vacuo

Following our previous derivation, let us consider a Lagrangian of a uniform medium with density  $\rho$ . As usual, the action is

$$\mathcal{S} \simeq \int d^4x \sqrt{g} \rho \quad (12)$$

with  $g \triangleq -\det g$  and  $d^4x \sqrt{g}$  being the (invariant) volume element. Following our previous procedure, we replace  $dx^\mu$  by its dimensionless ratio  $d\tilde{x}^\mu$  normalized by the Ricci length

$$dx^\mu \rightarrow d\tilde{x}^\mu = \frac{dx^\mu}{a_{\mathcal{R}}} = |\mathcal{R}|^{1/2} dx^\mu,$$

or

$$d^4x \rightarrow d^4\tilde{x} = d^4x \mathcal{R}^2. \quad (13)$$

The action becomes

$$\mathcal{S} \simeq \int d^4\tilde{x} \sqrt{g} \rho = \int d^4x \mathcal{R}^2 \sqrt{g} \rho. \quad (14)$$

Formally sending  $\rho$  to zero, we obtain the Lagrangian for gravitational field in vacuo

$$\mathcal{S}_{vacuo} = \int d^4x \sqrt{g} \mathcal{R}^2. \quad (15)$$

The procedure of setting the uniform matter density  $\rho$  to zero is of formality. We emphasize that, in our approach, gravitational field is not allowed to exist by itself, i.e., in isolation. It must always couple with matter, albeit this amount of matter might be negligible. The matter which plays the role of vacuo could be tentatively taken to be the zero-point energy background. We shall derive the full Lagrangian of gravity coupled with matter in Section 7 then show how to obtain  $\mathcal{S}_{vacuo}$  via a formality there.

The form of  $\mathcal{S}_{vacuo}$  is strongly restrained. It explicitly forbids the cosmological term, the Einstein-Hilbert term  $\mathcal{R}$ , as well as all other terms, except the quadratic term. Unlike the formal quadratic Lagrangian, though, the  $\mathcal{R}^2$  terms in our approach arose organically via the volume element. Each of the two terms  $\mathcal{R}^2$  and  $\sqrt{g}$  participates in action (15) for a legitimate reason, rather than an arbitrary fashion. Also, the resemblance between  $\mathcal{S}_{vacuo}$  and the quadratic Lagrangian is misleading; our full Lagrangian is not  $\mathcal{R}^2$  gravity when coupled with matter as we shall show in Section 7.

For the time being, it suffices to comment that the purely quadratic Lagrangian has been studied quite extensively in the past, see e.g. [10, 11, 14, 26–29]. For the sake of completeness, we nonetheless cite the results here. Upon the functional variation of  $g_{\mu\nu}$ , the field equations are

$$\mathcal{R} \left( \mathcal{R}_{\mu\nu} - \frac{1}{4} g_{\mu\nu} \mathcal{R} \right) + (g_{\mu\nu} \square - \nabla_\mu \nabla_\nu) \mathcal{R} = 0, \quad (16)$$

with  $\square \triangleq \nabla^\mu \nabla_\mu$ . Upon taking the trace

$$\square \mathcal{R} = 0, \quad (17)$$

the field equations in vacuo are simplified to

$$\mathcal{R} \left( \mathcal{R}_{\mu\nu} - \frac{1}{4} g_{\mu\nu} \mathcal{R} \right) = \nabla_\mu \nabla_\nu \mathcal{R} \quad (18)$$

which is a set of fourth-order PDEs as compared with the second-order Einstein field equations in vacuo

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = 0 \quad (19)$$

#### The well-posed Cauchy problem:

For a theory to have predictive power, it must yield a well-behaved evolution. This entails two issues: (i) The evolution be unique from a set of initial conditions, and (ii) The amount of information needed in the initial condition be manageable.

The well-posedness of the initial value problem for  $\mathcal{R}^2$  Lagrangian has been examined in literature [12, 13]. We only review its main features here. The purely quadratic Lagrangian  $\mathcal{R}^2$  is equivalent to a second-order tensor-scalar Lagrangian

$$\int d^4x \sqrt{g} \left( \phi \mathcal{R} - \frac{1}{2} \phi^2 \right) \quad (20)$$

with the unknown fields being the ten metric components  $g_{\mu\nu}$  and an auxiliary scalar field  $\phi$ . Upon functional variation of  $g_{\mu\nu}$  and  $\phi$ , the field equations are

$$\begin{cases} \phi (\mathcal{R}_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\phi) + (g_{\mu\nu}\square - \nabla_\mu \nabla_\nu) \phi = 0 \\ \phi = \mathcal{R} \end{cases} \quad (21)$$

The scalar field  $\phi$  happens to coincide with the Ricci scalar as a consequence of the field equations (21). The Cauchy problem is thus well-posed, with the Cauchy data consisting of the values of  $g_{\mu\nu}$ ,  $\partial_0 g_{\mu\nu}$ ,  $\mathcal{R}$ ,  $\partial_0 \mathcal{R}$  on the initial 3-hypersurface  $\Sigma$ . Normally one would expect a fourth-order theory to involve  $g_{\mu\nu}$ ,  $\partial_0 g_{\mu\nu}$ ,  $\partial_0^2 g_{\mu\nu}$ ,  $\partial_0^3 g_{\mu\nu}$  but the  $\mathcal{R}^2$  theory is special: although  $\mathcal{R}$  involves  $\partial_0^2 g_{\mu\nu}$ , and  $\partial_0 \mathcal{R}$  involves  $\partial_0^3 g_{\mu\nu}$ , its Cauchy data do not require full information of  $\partial_0^2 g_{\mu\nu}$  and  $\partial_0^3 g_{\mu\nu}$  but just a portion of it, the amount of information which is parsimoniously and neatly encoded in  $\mathcal{R}$  and  $\partial_0 \mathcal{R}$ . Being cast in this way, the evolution of the gravitational field involves two “positions” ( $g$  and  $\mathcal{R}$ ) and their “momenta” ( $\partial_0 g$  and  $\partial_0 \mathcal{R}$ ) rather than one “position” ( $g$ ) and its three lowest time-derivatives. The utility of  $\mathcal{R}$  as a “position” also nicely dovetails with its very meaning as the scale-setter for the Ricci length which is needed to specify the length scale for every point on the initial hypersurface  $\Sigma$ .

Regarding the amount of information needed in the initial condition, one major reason often cited in favor of the Einstein-Hilbert action  $\mathcal{R}$  is that it is a second-order theory and thus its Cauchy data advantageously avoid second- or higher-order derivatives w.r.t. time. In light of our discussion above,  $\mathcal{R}^2$  theory is not much more expensive than Einstein’s gravity; beside  $g$  and  $\partial_0 g$ , it only requires information regarding the values of the Ricci length and the first-order time-derivative of the Ricci length on the initial hypersurface  $\Sigma$ .

### Recovery of solutions to Einstein’s gravity:

As is well-understood, the field equation for  $\mathcal{R}^2$  gravity in vacuo accepts all solutions to the Einstein’s field equation in vacuo including the cosmological constant

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} + \Lambda g_{\mu\nu} = 0 \quad (22)$$

(which is equivalent to, upon taking the trace,  $\mathcal{R}_{\mu\nu} = \Lambda g_{\mu\nu}$ ). Thus the post-Newtonian phenomenology would be recovered if the corrections prove to be small and  $\Lambda$  takes on a small value at cosmic distances. Also note that the  $\mathcal{R}^2$  field equations (18) generally adopt solutions with  $\mathcal{R} \neq 0$ , thereby justifying the utility of the Ricci length  $a_{\mathcal{R}}$  in our theory.

The  $\mathcal{R}^2$  field equations in vacuo are sufficiently tractable; we shall provide two static spherically symmetric solutions in Sections (4) and (6). The  $\mathcal{R}^2$  Lagrangian admits a richer phenomenology. In particular it allows solutions with non-constant  $\mathcal{R}$ , a fertile ground for the scale adaptation to manifest. We shall present one such solution in Section 6 which has interesting implications in the physics of black holes. Yet even for constant- $\mathcal{R}$  solutions in vacuo, thanks to its extra boundary conditions, the solutions carry different meaning and application from Schwarzschild-de Sitter.<sup>8</sup> The rationale to study  $\mathcal{R}^2$  did not stem from its richness in phenomenology, however. Rather, the  $\mathcal{R}^2$  term arises organically via our replacement procedure. We note that in writing down his field equations, Einstein did not appeal to a principle but aimed at recovering Newtonian gravity. The Einstein-Hilbert action is simplest one that yields nontrivial phenomenology (with the action  $\int d^4x \sqrt{g}$  yielding no dynamics, for example). The main rationale to use the Einstein-Hilbert action lies with its second-order nature with its Cauchy data for Einstein gravity consists only  $g_{\mu\nu}$  and  $\partial_0 g_{\mu\nu}$ . This simplicity is not exclusive for Einstein gravity, however, but for  $\mathcal{R}^2$  gravity as well as we showed above.

With Einstein’s gravity began to show limitations to capture several recent problems, such as the dark matter problem, it is a usual practice to add new form of matter to the stress-energy tensor. Curvature-scaling gravity, on the other hand, provides a natural modification of the geometrical sector of the field equation. This could open a pathway to solve the dark matter problem without resorting to some hypothetical non-luminous matter. We shall explore this possibility in Section (4).

### Buchdahl’s solution for static spherically symmetric setup:

A fourth-order theory,  $\mathcal{R}^2$  gravity generally requires more boundary conditions than the second-order Einstein’s field equation. The static spherically symmetric case was examined in details over 50 years ago by Buchdahl [10]. We shall revisit his derivation, extend it and recast his solution in a more transparent way which helps illuminate the meaning of the terms in his solution. Our detailed derivation is presented in Appendix B. Here, we only summarize our finding.

<sup>8</sup> Solutions that deviate from the Einstein field equations with cosmological constant can be categorized in two groups. One group of solutions entails  $\mathcal{R} \neq 0$ . The other group of solutions still corresponds to  $\mathcal{R} = \text{const}$  but the extra boundary conditions introduce additional dimensional parameters (in terms of length). The latter group of solutions in general cannot be transformed into a constant- $\mathcal{R}$  solutions to the Einstein field equations with cosmological constant by an everywhere-differentiable coordinate transform. This point is important in our consideration of the dark matter problem in Section 4.

We deliberately cast the line element in the following form, where  $x$  is a radial coordinate variable ( $dx^0 \triangleq c dt$ ):

$$ds^2 = \frac{4}{\mathcal{R}(x)} \left\{ \frac{p(x)}{4} \left[ -\frac{q(x)}{4x} (dx^0)^2 + \frac{4x}{q(x)} dx^2 \right] + x^2 d\Omega^2 \right\} \quad (23)$$

in which  $\mathcal{R}$  is the Ricci scalar and  $p, q$  two auxiliary functions, satisfying the “evolution” equations:

$$\begin{aligned} \mathcal{R} &= \pm \Lambda e^{k \int \frac{dx}{xq}} \\ q_x &= (1 \mp x^2) p \\ p_x &= \frac{3k^2}{4x} \frac{p}{q} \end{aligned} \quad (24)$$

The most important conclusion from Buchdahl’s result is that spherically symmetric solutions to  $\mathcal{R}^2$  gravity in vacuo require four parameters. By virtue of the solution (23,24) above, the parameters are:

- $\Lambda$ , which measures the large-distance curvature. It is nothing but the de Sitter parameter. In Einstein gravity,  $\Lambda = 0$  unless one expressly introduces a cosmological constant. In  $\mathcal{R}^2$  gravity, a non-zero  $\Lambda$  can freely arise. It should be viewed as the boundary condition at  $x \rightarrow \infty$ , thus merging with the cosmological background at cosmic distances.
- $k$ , which allows the Ricci scalar to deviate from constant  $\Lambda$ . We shall call the deviation the anomalous curvature.
- $p(x_0)$  and  $q(x_0)$  at some  $x_0$  of user’s convenience. In the formulation presented above, the meaning of these two parameters is obscure.

In Sections 4 and 6, we shall utilize a more conventional choice of coordinates. As such, the parameters are no longer the same as shown in here, with their meanings becoming more self-evident, but the number of degrees of freedom remains 4.

## 4 Connection of curvature-scaling gravity to Mannheim's conformal gravity-based treatment of galactic rotation curves

In this section, we provide a specific solution in closed form for the static spherically symmetric setup. We next find a connection of our solution to a solution which Mannheim and Kazanas obtained in the theory of conformal gravity [2]. Based on their solution, Mannheim has advocated a detailed phenomenological theory for galactic rotation curves bypassing the need of dark matter [3–6]. As such, his theory opens a promising possibility to resolve the long-standing dark matter problem. From the connection of our solution to Mannheim-Kazanas's solution, we discuss the potential transition from curvature-scaling gravity to Mannheim's theory of galactic rotation curves.

### Our solution:

In general, the full solution for the static spherically symmetric setup is captured in (23) and (24). We are nonetheless interested in more explicit and closed-form solutions. Let us start with the metric in the more conventional coordinate setup

$$ds^2 = e^\alpha \left[ \Psi (dx^0)^2 - \frac{dr^2}{\Psi} - r^2 d\Omega^2 \right] \quad (25)$$

in which  $r \in (0, \infty)$  is the radial coordinate and  $dx^0 = c dt$ . The  $\mathcal{R}^2$  gravitational field equations for the  $tt$ - and  $\theta\theta$ -components are

$$\left( \mathcal{R}_{tt} - \frac{1}{4} g_{tt} \mathcal{R} \right) \mathcal{R} = -\Gamma_{tt}^r \mathcal{R}' \quad (26)$$

$$\left( \mathcal{R}_{\theta\theta} - \frac{1}{4} g_{\theta\theta} \mathcal{R} \right) \mathcal{R} = -\Gamma_{\theta\theta}^r \mathcal{R}' \quad (27)$$

which will be sufficient for the two unknowns  $\Psi$  and  $\alpha$ . We are able to find an analytical solution; the details of our calculation are presented in Appendix C. There are three equivalent representations for the solution. With the length element given in (25), the two unknowns  $\Psi$  and  $\alpha$  are:

- Representation I:

$$\begin{cases} \Psi &= (1 - 3ar_s) - \frac{r_s}{r} - \Lambda r^2 + a(2 - 3ar_s)r \\ e^\alpha &= (1 + ar)^{-2} \end{cases} \quad (28)$$

There are three parameters:  $\Lambda$ , specifying the large-distance curvature (the de Sitter parameter);  $r_s$ , the Schwarzschild radius taken as a free parameter (instead of  $\frac{GM}{2c^2}$ , a combination of  $G$ ,  $M$  and  $c$  separately); and  $a$  an additional parameter with unit of  $[\text{length}]^{-1}$ .

- Representation II:

$$\begin{cases} \Psi &= (1 - 3\beta\gamma) - \frac{\beta(2-3\beta\gamma)}{r} - \Lambda r^2 + \gamma r \\ e^\alpha &= \left( 1 + \frac{\gamma}{2-3\beta\gamma} r \right)^{-2} \end{cases} \quad (29)$$

This representation is a resemblance of Mannheim-Kazanas's solution in conformal gravity [2]. The  $\Psi$  term here is precisely the same term they obtained. Our solution is thus conformal to theirs via a conformal transformation to “gauge” away the  $e^\alpha$  term.

- Representation III:

$$\begin{cases} \Psi &= \sqrt{1 + \kappa} - \frac{r_s}{r} - \Lambda r^2 - \frac{\kappa}{3r_s} r \\ e^\alpha &= \left( 1 + \frac{1 - \sqrt{1 + \kappa}}{3r_s} r \right)^{-2} \end{cases} \quad (30)$$

$\kappa$  is an additional parameter in place of  $a$  in Representation I.

In all three representations, there appears a new term linear in  $r$  corresponding to a linear potential in addition to the Newton potential  $\frac{r_s}{r}$ . The main conclusion from our solution is that  $\mathcal{R}^2$  gravity gives rise to a modification of Newton's gravitational potential: This linear term arises from the geometrical structure (i.e., the metric) of spacetime itself, instead of an extra source of matter which one would usually add to the stress-energy tensor.

### Connection of curvature-scaling gravity to Mannheim's phenomenological theory for galactic rotation curves:

Over the last twenty years or so, Mannheim advocated a phenomenological theory to account for galactic rotation curves. The starting point of his approach is the static spherically symmetric solution Mannheim and Kazanas found in conformal gravity [2]. The metric they obtain, up to a conformal phase factor, is

$$ds^2 = \Psi dt^2 - \frac{dr^2}{\Psi} - r^2 d\Omega^2 \quad (31)$$

in which

$$\Psi = (1 - 3\beta\gamma) - \frac{\beta(2 - 3\beta\gamma)}{r} - \Lambda r^2 + \gamma r. \quad (32)$$

This solution is conformal to our solution presented above (see Representation II.) Beside the usual Schwarzschild term and the de Sitter term, their solution contains a linear term with regard to the radial coordinate. Mannheim subsequently argued that this new term arises from the boundary condition and the distribution details of the mass source. For sufficiently small  $\gamma$ , the metric recovers the well-known Schwarzschild-de Sitter metric. The  $\gamma r$  term presents an intriguing possibility to alter the Newtonian gravitational potential, thereby affecting the phenomenology of gravitation at larger distances, e.g. the galactic scales, while retaining the post-Newtonian predictions, i.e., the three classic tests of standard Einstein gravity if  $\gamma$  indeed has a small value. We refer the reader to the body of writing by Mannheim and collaborators [2–6, 8] for a comprehensive exposition of Mannheim's phenomenological treatment of galactic rotation curves. Here our objective is to build a bridge from curvature-scaling gravity to the Mannheim's theory.

To be concise, Mannheim's line of reasoning is that a fourth-order gravity theory (such as conformal gravity and curvature-scaling gravity) allows a richer set of solutions via its additional degrees of freedom. Classically, the Newton gravitational potential can be cast in term of the Laplace equation. In vacuo, it becomes the second-order Poisson equation (analogous to the Gauss law in electromagnetism). The radial part of the Laplacian operator in spherical coordinate

$$\nabla^2 V = \frac{1}{r^2} (r^2 V')' \quad (33)$$

which, interestingly, happens to equal to

$$\nabla^2 V = \frac{1}{r} (rV)'' ,$$

where prime denotes  $\partial_r$ . The Poisson equation

$$\nabla^2 V = 0 \quad (34)$$

thus has solution:

$$V = a - \frac{b}{r} \quad (35)$$

with  $a$  and  $b$  two integration constants. This static potential mimics the  $g_{00}$  component of the Schwarzschild metric,  $g_{00} = 1 - \frac{r_s}{r}$ . The radial force is thus

$$F = -\partial_r V = -\frac{b}{r^2}. \quad (36)$$

By the same token, the fourth-order ‘‘Laplacian’’ operator in spherical coordinate is

$$\nabla^4 V = \nabla^2 (\nabla^2 V) = \frac{1}{r} (r (\nabla^2 V))'' = \frac{1}{r} (rV)'''' . \quad (37)$$

The fourth-order Poisson equation

$$\nabla^4 V = 0 \quad (38)$$

thus adopts the solution

$$V = a - \frac{b}{r} + cr + \frac{d}{2}r^2 \quad (39)$$

with 4 constants of integration  $a$ ,  $b$ ,  $c$ ,  $d$ . This static potential mimics the  $\Psi$  term in Eq. 32. The corresponding centripetal force is

$$F = -\partial_r V = -\frac{b}{r^2} + c + dr. \quad (40)$$

The constant and linear forces, arising from the linear and quadratic terms respectively, bring in additional centripetal accelerations. That is to say, the  $\Psi$  term is a natural output of a fourth-order classical theory of gravitation.

Mannheim and collaborators proposed the following formula for the centripetal acceleration of a galaxy

$$\frac{v_{TOT}^2(R)}{R} = \frac{v_{LOC}^2(R)}{R} + \frac{\gamma_0 c^2}{2} - \kappa c^2 R \quad (41)$$

with  $R$  being the distance from the galaxy's center. The local galactic potential is computed in [3] to be

$$v_{LOC}^2(R) = \frac{N^* \beta^* c^2 R^2}{2R_0^3} \left[ I_0\left(\frac{R}{2R_0}\right) K_0\left(\frac{R}{2R_0}\right) - I_1\left(\frac{R}{2R_0}\right) K_1\left(\frac{R}{2R_0}\right) \right] + \frac{N^* \gamma^* c^2 R^2}{2R_0} I_1\left(\frac{R}{2R_0}\right) K_1\left(\frac{R}{2R_0}\right) \quad (42)$$

with  $R_0$  being the scale and  $N^* M_\odot = M$  the total mass of the galaxy. Formula (41) yields the asymptotic limit

$$v_{TOT}^2(R) \rightarrow \frac{N^* \beta^* c^2}{R} + \frac{N^* \gamma^* + \gamma_0}{2} c^2 R - \kappa c^2 R^2. \quad (43)$$

Beside the mass to light ratio being the only free parameter to be adjusted for each galaxy, these formulae involve 3 parameters  $\gamma^*$ ,  $\gamma_0$ ,  $\kappa$ , taken to be universal for all galaxies. In Formula (43), beside the usual falling off term (the  $1/R$  term), the linear and quadratic terms are expected to alter the behavior of the velocity curves at large value of the distance  $R$ . In [4–6] Mannheim and colleagues applied Formula (41) to a comprehensive set of 138 galactic rotation curves, a wide range of galaxies which cover different types of galaxies. They established good fit to the rotation curves and determined the values for the three universal parameters

$$\gamma^* = 5.42 \times 10^{-41} \text{cm}^{-1}, \quad \gamma_0 = 3.06 \times 10^{-30} \text{cm}^{-1}, \quad \kappa = 9.54 \times 10^{-54} \text{cm}^{-2}. \quad (44)$$

In their fits, the need for dark matter was avoided. The linear potential gives rise to a departure from Newton-Einstein at large distances, precisely in the range wherever the dark matter problem is encountered. With their values obtained in (44), the effects of  $N^* \gamma^*$ ,  $\gamma_0$  and  $\kappa$  only become as big as the Newtonian contribution at galactic scales. Mannheim thus concludes that the phenomenology overcomes the fine-tuning shortcomings of the dark matter-based framework. Since then, his theory has become an active field of research [15–25].

Given the impressive success of Mannheim's phenomenological treatment for galactic rotation curves with parsimonious assumptions, it would be interesting to see if curvature-scaling gravity can provide an alternative logical foundation – beside conformal gravity – for his theory. Like conformal gravity, curvature-scaling gravity is a fourth-order theory; therefore, the classical fourth-order Poisson equation argument repeated above is also applicable to it. Indeed, curvature-scaling gravity does accept a solution (25, 28, 29, 30) which is conformal to Mannheim-Kazanas's solution (31, 32) in conformal gravity. The particular potential function that Mannheim employs is not specific to the conformal gravity theory but rather is a generic feature, valid in curvature-scaling gravity as well. As such, Mannheim's theory of galactic rotation curves may not necessarily be an exclusive product of conformal gravity but it can arise from curvature-scaling gravity, or any fourth-order gravity in general. If this is the case, then Mannheim's phenomenological theory and results would be quite robust.

In recognition of Mannheim's endeavoring emphases on the importance of the linear  $\gamma r$  term in the behavior of galactic rotations, we shall call this term (in curvature-scaling gravity, that is) the Mannheim-Kazanas term, and  $\gamma$  the Mannheim-Kazanas parameter.<sup>9</sup> It is noteworthy that the  $\gamma r$  term will also appear in another solution that we shall derive in Section 6. The  $\gamma$  parameter will play an important role in the physics of black holes in the context of our curvature-scaling gravity in that section.

In passing, the extension of Mannheim's idea, essentially the linear Mannheim-Kazanas term playing a crucial role, to address the Pioneer anomalies has been studied recently [38–40]. Given that such a linear term is also present in our solution, we hope that curvature-scaling gravity could be of relevance to these studies as well.

### On the inapplicability of Birkhoff's theorem for conformal gravity and $\mathcal{R}^2$ gravity:

The metric we found in (25, 28, 29, 30) has a constant Ricci scalar. That is the reason why only 3 parameters are explicitly present in the metric. The fourth parameter which specifies the deviation of curvature from constancy is zero. As such, we have to address whether our metric is equivalent to the Schwarzschild-de Sitter metric which also has a constant Ricci scalar, according to Birkhoff's theorem.

It is well-understood that Birkhoff's theorem does not hold for high-order gravity; see [36], for example. Explicit solutions that violate Birkhoff's theorem for  $\mathcal{R}^{1+\delta}$  gravity have also been found in [37]. These solutions have non-constant Ricci scalar.

But even for metrics with a constant Ricci scalar, there are technical subtleties that prevent Birkhoff's theorem to be applicable for higher-order gravity. This is our focus in the rest of this section. To be concise, it is the extra boundary conditions present in a higher-order theory that require additional dimensional parameters (in terms of length) to specify them. These dimensional parameters cannot be transformed away by an everywhere-differentiable coordinate transformation.

<sup>9</sup> We must note that the spherical solution to conformal gravity had previously been obtained in [9].

More concretely, consider the metric (25, 28) in which  $r \in (0, \infty)$  and  $a > 0$ :

$$\begin{aligned} ds^2 &= e^\alpha \left[ \Psi dt^2 - \frac{dr^2}{\Psi} - r^2 d\Omega^2 \right] \\ \Psi &= (1 - 3ar_s) - \frac{r_s}{r} - \Lambda r^2 + a(2 - 3ar_s)r \\ e^\alpha &= (1 + ar)^{-2} \\ \Lambda &= \frac{\mathcal{R}_0}{12} + a^2(ar_s - 1). \end{aligned} \quad (45)$$

Let us tentatively make the coordinate change

$$r = \frac{r'}{1 - ar'}. \quad (46)$$

It is straightforward to verify that, in the new coordinate  $r'$ , the line element reads

$$ds^2 = (1 - ar')^2 \Psi dt^2 - \frac{d\rho^2}{(1 - ar')^2 \Psi} - r'^2 d\Omega \quad (47)$$

and that

$$(1 - ar')^2 \Psi = 1 - \frac{r_s}{r'} - \frac{\mathcal{R}_0}{12} r'^2 \quad (48)$$

Thus

$$ds^2 = \left( 1 - \frac{r_s}{r'} - \frac{\mathcal{R}_0}{12} r'^2 \right) dt^2 - \frac{dr'^2}{1 - \frac{r_s}{r'} - \frac{\mathcal{R}_0}{12} r'^2} - r'^2 d\Omega^2 \quad (49)$$

which is formally the Schwarzschild-de Sitter metric. However, this formality fails (thus rendering Birkhoff's theorem inapplicable for conformal gravity and curvature-scaling gravity) for four following reasons:

1. The coordinate transform is not differentiable everywhere. It is not analytic at  $r' = \frac{1}{a} > 0$ .
2. Related to Point (1), the mapping is not entire. To cover the range  $r \in (0, \infty)$ , the new coordinate only needs to cover  $r' \in (0, \frac{1}{a})$ ; namely, only a finite range  $(0, \frac{1}{a})$  is needed to label the space. The metric above would be seen as Schwarzschild-de Sitter if it held for the entire half axis. Since only a finite interval  $(0, \frac{1}{a})$  is mappable to the physical region  $r \in (0, \infty)$ , it does not approach the required limit as  $r' \rightarrow \frac{1}{a}$  from below. Say, for  $\Lambda' = 0$ , it is not asymptotically flat as  $r \rightarrow \infty$  (i.e.,  $r' \rightarrow \frac{1}{a}$  from below).<sup>10</sup>
3. The proof of Birkhoff's theorem is prejudicial in its choice of coordinates. It starts with the following choice:

$$ds^2 = A dt^2 - B dr^2 - r^2 d\Omega^2 \quad (50)$$

from which it deduces from the Einstein field equations that  $B = A^{-1}$ . This choice however expressly suppresses the freedom for the term  $a$  in metric (45) from being anything but zero, to start with. Therefore, it is not surprising that Birkhoff's conclusion is obtained. In Einstein gravity, this restriction poses no problem since Einstein gravity is a second-order theory. For fourth-order gravity such as conformal gravity or  $\mathcal{R}^2$  gravity, there are two more degrees of freedom in their static spherically symmetric solutions. In both theories, one of the additional parameters is  $a$  (or equivalently, the Mannheim-Kazanas parameter  $\gamma$ ). The other remaining degree of freedom is manifest only in  $\mathcal{R}^2$  gravity, however. It measures the deviation of curvature away from constancy – we shall call it the anomalous curvature, controlled by a new parameter  $\epsilon$ . The topics of anomalous curvature and  $\epsilon$  will be covered in Section 6.

4. The additional boundary conditions in higher-order gravity introduce new length scale to the solutions. In Einstein gravity, the freedom of scale is broken in the form of the Schwarzschild radius  $r_s$  (which is of unit of length). In  $\mathcal{R}^2$  gravity, two extra length scales appear in the form of  $a$  (or, equivalently,  $\gamma$ ) and  $\epsilon$  which are of dimension of  $[\text{length}]^{-1}$  and  $[\text{length}]^{-2}$  respectively.

In conclusion, it is not permissible to make a coordinate transformation to remove the Mannheim-Kazanas linear term,  $\gamma r$ , in either conformal gravity or curvature-scaling gravity. This term is not an artifact of coordinate choices; rather, it is set by boundary conditions, and has physical and detectable effects on the motion of matter in gravitational field when it is present<sup>11</sup>.

<sup>10</sup> We thank Mannheim for his affirmation of these technical points, which himself and O'Brien utilized in [4–6].

<sup>11</sup> It will be interesting to explore how this conclusion affects gravitational waves since Birkhoff's theorem no longer applies for  $\mathcal{R}^2$  gravity. This question has been raised in [36], for example.

## 5 Logical inferences from the postulate of Ricci scalar as dynamical scale-setter

### 5.1 The anisotropy in time scaling

In a recent proposal put forward by Hořava [1], time is treated anisotropically. In Hořava-Lifshitz gravity the time direction, unlike the spatial directions, acquires a dynamical exponent  $z = 3$  rather than 1 in its scaling in the Wilsonian renormalization group process. The anisotropy is invoked to deal with the inherent asymmetry of the role of time vs. space in the dynamics of physical processes. The Lorentz symmetry is expressly broken in the ultraviolet limit, and is expected to emerge in the infrared limit, thus restoring the space-time isotropy in the classical limit.

Interestingly the anisotropy of time vs. space also arises quite naturally in our approach. Consider the Schrödinger equation for the Hydrogen atom

$$i\hbar \frac{\partial}{\partial t} \Psi = -\frac{\hbar^2}{2m_e} \nabla^2 \Psi - \frac{e^2}{r} \Psi. \quad (51)$$

Let us recall that in Section 1, Postulate (I) mandates that the form of Eq. (51) be the same in all pockets of spacetime, whereas Postulate (II) requires that the length scale in Eq. (51) be pegged to the local Ricci length, defined as  $a_{\mathcal{R}} \triangleq |\mathcal{R}|^{-1/2}$ . Consequently, these two conditions mean that Eq. (51) is invariant when expressed in terms of dimensionless ratios of space and time using the Ricci length as denominator. More concretely, rewriting the coordinate differentials as

$$\begin{cases} d\vec{x} &= a_{\mathcal{R}} d\vec{\tilde{x}} \\ dt &= a_{\mathcal{R}}^\eta d\tilde{t} \end{cases} \quad (52)$$

in which we introduce an anomalous scaling exponent  $\eta$  for time, the Schrödinger equation (51) becomes

$$i \frac{\hbar}{a_{\mathcal{R}}^\eta} \frac{\partial}{\partial \tilde{t}} \Psi = -\frac{\hbar^2}{2m_e a_{\mathcal{R}}^2} \tilde{\nabla}^2 \Psi - \frac{e^2}{a_{\mathcal{R}} \tilde{r}} \Psi, \quad (53)$$

or

$$i \left( \hbar a_{\mathcal{R}}^{1-\eta} \right) \frac{\partial}{\partial \tilde{t}} \Psi = -\frac{\left( \hbar a_{\mathcal{R}}^{-1/2} \right)^2}{2m_e} \tilde{\nabla}^2 \Psi - \frac{e^2}{\tilde{r}} \Psi. \quad (54)$$

The invariance requirement alluded above then enforces two conditions that

$$\begin{cases} \hbar a_{\mathcal{R}}^{1-\eta} &= \text{const} \\ \hbar a_{\mathcal{R}}^{-1/2} &= \text{const} \end{cases} \quad (55)$$

Equating the exponents of the two questions, we obtain

$$\eta = \frac{3}{2}$$

which means an anisotropy in the scaling for time interval

$$dt \propto a_{\mathcal{R}}^{3/2} \propto |\mathcal{R}|^{-3/4}. \quad (56)$$

This conclusion is not limited to the nonrelativistic Schrödinger equation for the Hydrogen atom. Rather, it holds generically for any physical system. In particular, it holds for the partition of QED which encompasses the (relativistic) Dirac equation and the Maxwell equations of electromagnetism, as shown in Appendix A. The emergence of time anisotropy is a conceptual conclusion derived solely from Postulates (I) and (II). The time anisotropy is inherent in all physical laws established to date.

As the observer moves from one local region to the next on the manifold, the Ricci length varies (together with the metric  $g_{\mu\nu}$ ), but the observer would not be able to detect anything amiss using objects, rulers and clocks within his surroundings as reference. Everything in his surroundings scales in exact proportion, and every clock in his surroundings speeds up or slows down in sync. For example, if he is to conduct the Michelson-Morley experiment, he will obtain a unique value for the speed of light regardless of his state of motion and/or the direction of the light beam. Wherever he sits on the manifold, the laws of physics look the same; it is only his ruler and clock and all objects around him (including his own body size and his own heart rate) adjust to the prevailing value of the Ricci scalar. The observer, however, in principle can detect the discrepancies between his region and another region by exchanging light signals with an observer residing in the other region. We thus conclude:

#### The scaling of time duration:

*The time duration of physical processes (i.e., the inverse of the oscillatory rate of clocks) at a given point on the spacetime manifold obeys the scaling rule  $dt \propto a_{\mathcal{R}}^\eta$  with  $\eta = \frac{3}{2}$  and  $a_{\mathcal{R}}$  being the local Ricci length, defined as  $a_{\mathcal{R}} \triangleq |\mathcal{R}|^{-1/2}$ .*



## 5.2 The first conceptual departure

The utility of the Ricci scalar as the local scale-setter for physical laws leads to several conceptual departures from the conventional belief. By the very nature of the equivalence principle, physical laws are valid locally; as such, it is conceivable that the parameters of the physical laws are also valid only locally. One of the conceptual casualties we shall encounter is the abandonment of  $\hbar$  as a universal constant. Whilst the laws of physics – say, the Schrödinger equation – retain their forms in each individual local region, the parameters – in this case,  $\hbar$  – which keep track of the laws need not be universal. This can be seen directly from the conditions (55) above that

$$\hbar \propto a_{\mathcal{R}}^{1/2} = |\mathcal{R}|^{-1/4} \quad (57)$$

namely, the value of the Planck constant depends on the prevailing Ricci scalar. At first this result looks alarming enough. After all, it is an undisputable fact that  $\hbar$  is a fundamental constant which ubiquitously governs different branches of physics – from photons and quarks in particle physics, to phonons, magnons, quantum Hall effect in condensed matter physics, to the periodic table in quantum chemistry, to the alpha decay and the shell model in nuclear physics, to name a few. For all these diversified branches of physics, it has been established beyond doubt that there is one single value of  $\hbar$ .

Our approach unequivocally preserves the all-embracing domain of applications of quantum physics. The Planck constant retains its sovereignty over all quantum physical processes that occur in each individual local region. Yet it is in no contradiction with the conceivability that its value is valid only locally to each region. Moreover, it is imperative to realize that the ubiquity in the value for  $\hbar$  in all these branches of physics has been based on measurements conducted in the surrounding region of the Earth<sup>12</sup>. Since  $\hbar$  does govern wide-ranging physics in a region – i.e., our region – it was natural to generalize its value to everywhere else on the spacetime manifold. This extrapolation in the value of  $\hbar$  is overreaching and unnecessary theoretically, and has not been falsified experimentally.

The Planck constant thus should not be taken for granted to be an omnipotent fundamental; rather, in each local pocket on the manifold,  $\hbar$  should be allowed to adapt to the prevailing value of the Ricci scalar  $\mathcal{R}$  in the precise relationship (57),  $\hbar \propto |\mathcal{R}|^{-1/4}$ . Note that this is not a triviality of choosing a set of measurement apparatus at the observer's discretion – an idea often circulated in literature<sup>13</sup>. The ratio of the Ricci scalar at two different locations is an objective measure, and thus the ratio of in values of  $\hbar$  at two different locations is also an objective measure without being subject to any arbitrary choice of ruler and clock at the observer's disposal. The abandonment of the universality in the value of  $\hbar$  does no violence to any physical principles of special relativity, the equivalence principle, the general covariance principle, as well as quantum physics.

There is a more intuitive way to obtain the scaling rule (57) for  $\hbar$ . At first the astute reader might have noticed that in Einstein's gedanken elevator, the size of every object should be determined by quantum mechanics via, say, the Schrödinger equation which governs its atoms. For example, the Bohr radius  $r_{Bohr} = \frac{\hbar^2}{m_e e}$  of the Hydrogen atom should be fully fixed by constants of Nature, such as  $\hbar$ . Equipped with this mindset, the reader may object that the size of an object is complied with  $\hbar$  rather than the Ricci scalar  $\mathcal{R}$  per our Postulate (II). Our approach, however, is a radical departure from the orthodox view: we hold that the Ricci length is of a more fundamental status than is the Bohr radius. The Schrödinger equation retains exactly the same form in every local spacetime region that the elevator falls through. Yet, the Bohr radius – and equivalently  $\hbar$  – is pegged to the Ricci length, and thus is allowed (indeed required) to vary together with  $\mathcal{R}$  from point to point on the manifold. With  $m_e$  and  $e$  taken as intrinsic quantities of the electron, given that  $r_{Bohr}$  is pegged to  $a_{\mathcal{R}}$  (i.e.,  $r_{Bohr} = \frac{\hbar^2}{m_e e} \propto a_{\mathcal{R}}$ ) one can immediately deduce that  $\hbar \propto a_{\mathcal{R}}^{1/2}$  in agreement with the scaling rule above. The (re-)assignment of fundamental role from  $\hbar$  to  $\mathcal{R}$  in setting the length scales for physical laws also has a conceptual appeal: spacetime should provide the length scale against which non-gravitational processes are measured, rather than the other way around.

Although the size of physical objects and the beating rate of clocks can vary from one local region to the next on the manifold, the observer cannot simply bring identical objects from two different regions together side-by-side to catch out their relative difference. If he manages to do so, the objects would adjust their sizes to the new Ricci length where they sit and no relative difference in size would manifest. Likewise, he cannot simply bring synchronized clocks from two different regions together side-by-side to catch out a mismatch in their oscillatory rates. If he manages to do so, the clocks will adjust their vibrations to the new local scale and click in sync. To detect objective discrepancies between two

<sup>12</sup> One might object that light from distant galaxies and supernovae appears to support a universal value of  $\hbar$  as well. We shall dispense with this point momentarily in the next two subsections (5.3) and (5.4).

<sup>13</sup> This issue has been misinterpreted, if not misunderstood, in various papers in literature. One favorite critique pretends that a dimensional  $\hbar$  can be made to take any value at the observer's will, by changing the units of length and time; e.g., see [42]. Although handy, this argument is naïve and false for at least three reasons. First, in our approach, the rulers are prepared identical, and the clocks synchronized. The ratio in the values of  $\mathcal{R}$  at two different locations is an objective dimensionless number; so is the ratio in the values of  $\hbar$  at those said locations regardless of the choice of units. Second, the same critique must also dismiss the Lorentz contraction and time dilation as a matter of unit choice since length and time are dimensional. This conclusion is wrong; time dilation is a real effect detectable in the lifetime of muons created in the upper stratosphere yet able to reach the Earth's sea level. Third, when this sort of critique dismisses the non-universality for  $\hbar$  as a matter of units, it simultaneously rejects the very meaning of the universality of  $\hbar$  which these authors deem sacred (Wouldn't the equality of  $\hbar$  at different locations be a matter of unit choice too?) As such, this self-defeating critique should be exorcised from the scientific vocabulary.

regions, the observer must compare the wavelengths and frequencies of light signals exchanged between the two regions. This procedure is not unlike the standard practice astronomers use to detect the redshift of light from distant galaxies. Once again, in our illustration so far, the rulers are identical, and the clocks synchronized. The objective and relative differences in rulers' size and clocks' rate arises from the difference in the values of the Ricci scalar in the two locations.

The anisotropic time scaling leads to another conceptual departure which is our focus in the following section.

### 5.3 The second conceptual departure and the preservation of causality

Let us first review the essence of the Michelson-Morley experiment. The Michelson-Morley interferometer consists of two light beams, one beam – the “longitudinal” arm – traveling along and against the Earth's direction of motion in the solar system, whereas the other beam – the “transverse” arm – traveling perpendicularly to the Earth's motion. The two beams are then made to meet and interfere. The two beams were initially expected by Michelson and Morley to accumulate different travel times because of their different alignments with regard to the Earth's motion. As such, they were expected to experience a relative phase shift and thus should produce an interference pattern. The result that Michelson and Morley obtained was null; no interference was found. Since then, the Michelson-Morley finding has been reconfirmed with increasing accuracy. Their finding is interpreted in special relativity as solid evidence in support of a constancy in  $c$ , that light travels at the same speed regardless of its direction, whether along or against or transverse to the Earth's motion in the solar system.

We hold the Michelson-Morley experimental finding and Einstein's theoretical conclusion of the constancy of  $c$  as well-established truths, and we shall enforce them as venerable facts in our approach. Nonetheless it is imperative to realize that the Michelson-Morley experiment only confirms the common value of  $c$  for light beams that travel at the same location, viz. within their interferometer stationed in Cleveland. If we are to bring their interferometer to, say, the city of New York (NYC), and repeat their experiment at the new place, their null finding and thus a common value of  $c$  for the two light beams within the interferometer now residing in NYC should not be in question. However, it is perfectly legitimate to question the equality between the Cleveland-bound value of  $c$  and the NYC-bound value of  $c$ . The Michelson-Morley interferometer has nothing to say as of whether or not the values of  $c$  encountered at Cleveland and at NYC are identical; it was not designed to answer that question. The Michelson-Morley experiment commands no authority whatsoever over the equality (or the lack thereof) of the values of  $c$  measured at different locations. Indeed there has never been any experimental or observational evidence in support of a universal value for  $c$  at different locations in spacetime<sup>14</sup>. This is a subtle yet significant point which has been underappreciated in the development of relativity.

The importance of this observation of ours cannot be overstated. Imagine a series of observers  $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_n$  aligned on a light beam emitted from a light source. Let each observer separately measure the speed of the light beam as it passes by his location. Regardless of the motion of Observer  $\mathcal{O}_k$  (whether he runs toward the light source, away from it, or sideways), he must record a unique value  $c_k$  for the light speed at his location. This is the requirement of the constancy of the speed of light which we unconditionally enforce in our theory. Yet, the Michelson-Morley finding has nothing to say about the equality (or the lack thereof) for the members in the serial  $\{c_1, c_2, \dots, c_n\}$ . Each element in the series can depend on the state of spacetime where it is measured. In curvature-scaling gravity, the said state of spacetime is the Ricci scalar.

Let us prepare two identical replicas of the Michelson-Morley interferometer and a pair of synchronized clocks at our location, denoted by  $A$ , then send one replica and one clock to a remote location  $B$ . Let us assume that the two locations have different values for the Ricci scalar, such that  $\mathcal{R}(B) = \frac{1}{4} \mathcal{R}(A)$ . The Ricci lengths at the two locations are related as  $a_{\mathcal{R}}(B) = 2 a_{\mathcal{R}}(A)$ . The  $B$ -interferometer adapts its size accordingly; i.e., the length of its two arms scale up by a factor of 2. At the same time, the  $B$ -clock which measures the vibration rate of atoms and light beams at  $B$  scales anisotropically per the time scaling rule (56): it requires an increase of  $2^{3/2}$ -fold for the atoms at  $B$  to make a full vibration as compared with the atoms residing at  $A$  to complete a full cycle. That is to say, the  $B$ -clock beats at a rate  $2\sqrt{2}$ -fold slower than the synchronized  $A$ -clock. The net result is that light travels at  $B$  at a slower speed compared with the light speed at  $A$ . This is because it requires a  $2\sqrt{2}$ -fold more amount of time for light at  $B$  to cover twice as long a distance, thus:  $c(B) = \frac{1}{\sqrt{2}} c(A)$ . So, each interferometer separately still registers a common value for  $c$  at its respective location – that is to say, each individual interferometer replicates the Michelson-Morley result at its respective location – yet the interferometers are free to take different values of  $c$  relative to each other. This is the second conceptual casualty being deduced from our Postulates (I) and (II). The value of  $c$  depends on the prevailing value of the Ricci scalar where the light beam is measured according to the following scaling rule:

$$c \propto a_{\mathcal{R}}^{-1/2} = |\mathcal{R}|^{1/4}. \quad (58)$$

For the observers mentioned above, the values of light speed they obtain are thus dependent on the values of the Ricci scalar where they are located,  $\{c_1, c_2, \dots, c_n\} \propto \{|\mathcal{R}_1|^{1/4}, |\mathcal{R}_2|^{1/4}, \dots, |\mathcal{R}_n|^{1/4}\}$ . This is, again, not a triviality in the

<sup>14</sup> There have been repeats of the Michelson-Morley experiment using star-light sources. But this detail is irrelevant since in this type of experiments the two light beams still met at the same location on Earth and the common value of  $c$  was obtained in the Earth-based lab. This type of experiments have nothing to say about the values of  $c$  at the respective locations of the two light sources.

choice of rulers and clocks at the observers' disposal<sup>15</sup>. The ratio  $\mathcal{R}_1/\mathcal{R}_2$ , e.g., is an objective measure of the spacetime manifold; as the result, the ratio  $c_1/c_2$  is also an objective measure.

The locality nature of the Lorentz symmetry, the relativity principle, and the value of  $c$  is enshrined in the very content of the equivalence principle. The equivalence principle mandates that special relativity hold locally in the tangent frames of each given point on the manifold. It is not meaningful whatsoever to talk about a global scope for Lorentz invariance, the relativity principle, and – as we now see – the value of  $c$  in a curved spacetime, although it is meaningful to do so for a Minkowski spacetime. (On the other hand, causality, i.e. the order in precedence of causally-connected events, is global in all configurations of spacetime as we shall show right below.) Together with  $\hbar$ ,  $c$  continue to oversee the physics within each local region:  $\hbar$  ubiquitously measures the strength of quantum effects in different branches of physics, and  $c$  controls the Lorentz symmetry. Yet, under no premises does this feature prevent them from acquiring a new value from one location to the next. Forcing them to have a universal value – a custom taken for granted in conventional wisdom – is an overreaching and unnecessary practice, if not an unfulfillment of the spirit of the equivalence principle.

The lack of experimental support for a universal  $c$  notwithstanding, the reader might insist: Could there still be an underlying theoretical reason which might otherwise enforce the equality in the value of  $c$  for all locations? In particular: (i) Would the causality principle be the justifying agent, perhaps? (ii) Or, how then would a variability of  $c$  manage to avoid the troublesome superluminality? (iii) Last but not least, wouldn't a variability of  $c$  in spacetime directly violate the Lorentz symmetry? Ubiquitously, the equality in the value of  $c$  for all locations has been prejudicially rationalized out of the fears of an endangerment of causality and, related to it, an encounter of faster-than-light travel. These fears are unwarranted as we shall show momentarily. Our theory strictly forbids superluminality and strictly protects causality.

The proof of causality was provided in section Causality on page 8. For a pair of events no matter how far apart they are on the manifold, as long as they are causally connected, the order of their precedence is strictly protected. The proof only relies on the Lorentz invariance for each individual local frame; it does not depend on whether or not the speed of light is uniform along the timelike trajectory connecting the two events. In the series of events

$$t_A = t_1 < t_2 < t_3 < \dots < t_{n-1} < t_n = t_B \quad (59)$$

although the differences  $t_2 - t_1$ ,  $t_3 - t_2$ , ...,  $t_n - t_{n-1}$  are dependent on the coordinate choice, their signs are strictly positive. No coordinate transformation can frustrate the order of precedence (59). Timelike- and spacelike- trajectories do not mix. Null geodesics remain null geodesics in all coordinate choices. In all global coordinate systems, null geodesics strictly separate the set of timelike paths from the set of spacelike paths. No coordinate change can transform a spacelike path into a timelike path or vice versa. Another way to view the denouement is that since our theory only employs invariants (including  $c$  itself which – albeit a variable – is a Lorentz invariant thanks to its one-to-one correspondence with the (invariant) Ricci scalar,  $c \propto |\mathcal{R}|^{1/4}$ ), causality is automatically ensured.

Preceding the lack of universality in light speed<sup>16</sup>, the fears of violation of the Michelson-Morley result and of causality have been misplaced:

1. The speed of light at a given point and a given instant is the same for all light beams that pass through that given point at that given instant regardless of the directions of the light beams, viz. whether they are traveling along or against the Earth's motion, for example. By the same token, regardless of the motion of the observer, whether he travels along or against a light beam, he will record one common value for  $c$  for the light beam at his location. Point (1) is the very content of Einstein's postulate of constant light speed and of the Michelson-Morley's experimental conclusion. These results are fully retained in our approach.
2. The speed of light at a given point and a given instant is the maximum velocity that any physical object can attain when it travels across that given point at that given instant. Point (2) is a direct outcome of the Lorentz symmetry – a combination of Einstein's postulate of constant light speed and his relativity principle. This conclusion is also fully preserved in our approach. No objects could surpass light at any point in space and at any instant in time. At any given point on the manifold, light provides the upper bound of speed for all objects. The fact that the upper bound of speed is variable from one point to the next is irrelevant. Superluminality is strictly forbidden. As such, causality is strictly preserved.

Another misplaced objection raised above is that a variability of  $c$  would necessarily endanger Lorentz invariance. This objection is false. The Lorentz symmetry would be in jeopardy only if  $c$  is an explicit function of spacetime coordinates. In our approach,  $c$  is not a function directly of spacetime coordinates. Rather, it is solely a function of the intrinsic Ricci scalar which acquires a dynamics via that of  $g_{\mu\nu}$ . At each point on the manifold, via (58) the Ricci scalar appoints the value for  $c$  which then enters the length element for the local region enclosing the point. The length element possesses the Lorentz symmetry. As such, Lorentz invariance is enforced locally. We stress that  $c$  (and  $\hbar$  too) are not auxiliary fields that live on the manifold. They do not have a dynamics on their own right. That is to say, there are no terms such as  $\partial_\mu c$  or  $\vec{\nabla} \hbar$  present in our theory.

<sup>15</sup> See the related comment in footnote 13 on page 17.

<sup>16</sup> It would be apt to instead say “the loss of universality in light speed” from the historical perspective.

We attribute the conundrum to confusion in nomenclature: “constancy” versus “universality”. Whereas “constancy” means a common value for  $c$  at a given location regardless of the state of observer and/or emitter (for example, regardless of the direction of the light beam along or against the Earth’s motion and/or regardless of whether the observer is running along or against the light beam), “universality” on the other hand means a common value for  $c$  measured at different locations. The “constancy” of  $c$  by no mean implies its “universality”. Often, though, the two concepts are automatically paired hand-in-hand out of fear to safeguard causality and ensure the Michelson-Morley result everywhere on the manifold. The fear and practice are unwarranted; it is permissible to construct a manifold – as is done in our approach – which fully respects causality while preserving the constancy for  $c$  everywhere (i.e., ensuring the Michelson-Morley result at every individual location) yet disowning its universality. Moreover, from the constancy of  $c$ , any generalization to a universality for  $c$  has been an overreaching act. The statement often taken for granted that  $c$  takes the same value everywhere does not reflect an experimentally verified fact, but a presumption. To be upheld with scientific value, the assertion that the speed of light is universal needs be falsified. There have not been any experiments that falsify it or even attempt to falsify it. The Michelson-Morley setup did not. Neither have any experiments to date. Whereas the constancy-to-universality generalization is valid for special relativity since all points in a Minkowski spacetime are equal (no point in a flat spacetime holding a privileged position over the rest), the generalization fails for general relativity in which spacetime is inherently curved with local regions no longer being identical. Spacetime points are not created equal – the Ricci scalar in principle varies from point to point. General relativity breaks spacetime manifold into pockets each with its own scalar curvature and thus with – per our postulate on the role of the Ricci scalar – its own physical parameters, with  $c$  and  $\hbar$  being the most notable. The overreaching generalization then is a relic of special relativity, a relic that needs be repudiated in a general theory of spacetime <sup>17</sup>.

Considering the perplexity of the Michelson-Morley finding compared with one’s daily life’s “common sense”, as soon as the Michelson-Morley conclusion crystallized in special relativity, it has been a safe practice to superficially extrapolate its (correct) conclusion of “constancy” for  $c$  to an (overreaching) speculation of “universality” for  $c$  and subconsciously hold the (unverified) presumption as an unassailable truth. And it has been an equally safe practice to disregard any serious attempt to challenge such a misperception as unworthy of healthy discussions. Historical accounts indicate that, as of 1912, Einstein came to recognize the locality nature of special relativity and of the equivalence principle. Only in his later development of general relativity, in a frantic race against Hilbert, did he adopt a pragmatist’s route without a thorough pursuit or exploitation of the local – instead of global – validity of these principles.

In summary, we have arrived at the following depiction of the spacetime manifold: it constitutes a patchwork of local pockets of spacetime, each pocket obeying the special relativity (and all laws of non-gravitational origin such as quantum mechanics) yet each adopting its own length scale. The Ricci scalar  $\mathcal{R}$  determines the length scale for each individual local pocket. As one moves from one pocket to the next on the manifold, the form of physical laws remains unchanged but the beating rate of clocks and the size of rulers and every object (including the observer’s own body size and heart rate) must adapt to the prevailing value of  $\mathcal{R}$ . The fundamental constants  $\hbar$  and  $c$  continue to oversee the established physics in each local pocket but they too must adapt to the prevailing value of  $\mathcal{R}$ .

Compared with the logical conclusion of  $\hbar$ -variability, the relegation of the speed of light from its once-sacred position in physics is far more alarming. Yet, both of the variabilities in  $c$  and  $\hbar$  are natural ramifications of our Postulate (II) proposed in Section 1 that the Ricci length is of a more fundamental status than all other lengths. Mathematically, their variabilities arise from the anisotropy of time scaling (56) which takes place in  $3 + 1$  dimensions. The Planck constant and speed of light are not God-given prefixed fundamentals; rather, their values are determined by the Ricci scalar in the spacetime pocket in which they are measured in a one-to-one correspondence:  $\hbar \propto |\mathcal{R}|^{-1/4}$ ,  $c \propto |\mathcal{R}|^{1/4}$ . We must also emphasize once again that despite their variabilities,  $\hbar$  and  $c$  are not auxiliary fields that live on the manifold <sup>18</sup>.

Lastly, let us compare our approach with Hořava’s proposal [1]. In Hořava’s approach, Lorentz invariance is assumed to emerge in the coarse-graining procedure in which fast-mode fluctuations are step-by-step integrated out. At the microscopic ultraviolet limit, Lorentz invariance is absent. In our approach, Lorentz invariance is fully preserved: the anisotropy in time scaling  $dt \propto a_{\mathcal{R}}^{3/2}$  is compensated by the  $c$ -variability,  $c \propto a_{\mathcal{R}}^{-1/2}$  such that  $dx^0 \triangleq c dt \propto a_{\mathcal{R}}$  exactly as its spatial counterparts  $d\vec{x} \propto a_{\mathcal{R}}$ .

## 5.4 Applications of the anisotropic time scaling in cosmology: An invitation

The logical conclusion of  $c$ -variability in the preceding section has very far-reaching consequences. If we are to repeat the Michelson-Morley experiment in Cleveland today, it is not in question that we shall reproduce their null finding for

<sup>17</sup> In getting acquainted with special relativity, the students are ubiquitously familiarized with popular examples of Einstein’s light-years-long trains and twins taking light-years-long journeys. These examples have not helped; indeed they sowed the seed of a misperception that the speed of light must be uniform along all the way from Earth to distant stars light years away even in curved spacetime. We need to exorcise this misconception and correct the unfortunate course of history.

<sup>18</sup> This conclusion of ours is in contrast to the approach Moffat, Magueijo and others subscribed to in [52–55]. These authors attempted several mechanistic assignments of a dynamics for  $c$ . In light of our approach, such assignments were unnecessary. If the reader nonetheless insists on a dynamics for  $c$  and  $\hbar$ , the reader may view the dynamics of  $\mathcal{R}$ , i.e. the field equations for  $g_{\mu\nu}$ , as the underlying dynamics for  $c$  and  $\hbar$  since these parameters are one-to-one related to  $\mathcal{R}$ .

the interference pattern, and that we shall unequivocally concur on a common value of  $c$  for the two light beams in our interferometer that exist today. Yet nothing in principle – viz., nothing amongst the causality principle, the relativity principle, Lorentz invariance, the equivalence principle, or the general covariance principle – compels an equality between the value of  $c$  that we encounter in Cleveland today and the value of  $c$  that Michelson and Morley experienced in Cleveland back in 1887. According to our conclusions deduced in the preceding section, the value of  $c$  in a local spacetime pocket is pegged to the Ricci scalar in the pocket per the scaling rule (58):  $c \propto |\mathcal{R}|^{1/4}$ , and thus if the Ricci scalar in the local region has undergone a change in value, then a change in the value of  $c$  must follow. Obviously we do not expect Cleveland’s surroundings to have expanded during the last 126 years (since galaxies resist cosmic expansion), yet our Cleveland metaphor helped illustrate our logic.

The cosmos, however, is an ideal laboratory in which the  $c$ -variability would show its glory. With the universe undergoing an expansion for several billions years (i.e., its Ricci scalar having been steadily decreasing) [45], light emitted from stars at a distant past has been traveling through a succession of local spacetime pockets with steadily decreasing value of  $c$ . At any given point on its transit toward Earth, the photon lives in a local tangent frame with special relativity in full effect but the value of its speed progressively adapts to the prevailing value of  $\mathcal{R}$  where the photon passes by. Over the life of the universe,  $c$  has been falling by several orders in magnitude, rendering qualitatively detectable and quantitatively measurable effects in observational cosmology. This conceptual point holds the key to resolving several pressing problems in standard cosmography and cosmology. To be concise, the standard paradigm of cosmology, together with the Friedmann model as its foundation, overlook the progressive decrease in light speed in its treatment of the Hubble law, the redshift-distance relationship, the interpretation of high- $z$  objects (Type Ia supernovae), and the theoretical difficulties with regard to the horizon, flatness, and cosmic coincidences. A full-fledged excursion on the consequences of the  $c$ -variability in cosmography and cosmology shall be presented in Sections 8 and 9. In what follows, we shall give a brief outline as of how the scaling rule (58) alone will help resolve four outstanding problems: the age problem, the interpretation of Type Ia supernovae (while bypassing the accelerating expansion), the horizon problem, and the flatness problem (while bypassing the inflationary expansion.) The results presented below are model-independent. They directly follow from the scaling rule (58),  $c \propto |\mathcal{R}|^{1/4}$  without any resort to any specific model of matter distribution (such as a modification to the Friedmann model, for example.)

Imagine a distant galaxy which emitted a beam of photons. On their journey toward the Earth, the photons went through a succession of spacetime pockets with increasingly larger cosmic scale factor  $a$ , i.e., decreasing scalar curvature  $\mathcal{R}$ . (Also, note that the cosmic scale factor  $a$  is proportional to the Ricci length  $a_{\mathcal{R}}$ , so we shall work with  $a$  in what follows.) The photon wavelength gets “stretched out”, thus the photons getting “redshifted”, a fact that astronomers agree upon. The standard formula for the redshift  $z$  is (see Eq. (111)):

$$1 + z = a^{-1} \quad (60)$$

with  $a$  being the cosmic factor at the moment the photons were emitted (with the current cosmic factor set equal 1.) However, this formula – as well as all other formulae in standard cosmology – neglect the effect of a steadily falling light speed, an effect which must be taken into account. This effect modifies the redshift formula to (see Eqs. (118) and (360)):

$$1 + z_{\text{observed}} = a^{-3/2} \quad (61)$$

where  $z_{\text{observed}}$  is the redshift value that appears in the astronomer’s telescope (see Section 8.2.2 and Appendix G for derivation of this new formula.) The exponent of  $\frac{3}{2}$  directly stems from the anisotropic exponent in time scaling  $\eta = \frac{3}{2}$ <sup>19</sup>.

#### 1. The age problem:

Given that the cosmos was smaller in the past,  $a < 1$ , the new redshift formula (61) means that the actual redshift is enhanced:  $z_{\text{observed}} > z$ . That is to say, if the astronomer unknowingly uses the standard redshift formula (60) to back out the value of the cosmic factor  $a$  from  $z_{\text{observed}}$ , she would inadvertently overestimate the value of  $a$ . The overestimate is consistent for light sources at all distances. Therefore, when expanding the redshift formula for low- $z$  objects, the Hubble law is inadvertently missing an extra multiplicative factor of  $\frac{3}{2}$  (see Section 8.3 for details.) That is to say, the estimated value obtained from the regression of  $z_{\text{observed}}$  versus the distance  $d$  of the objects is not the actual Hubble constant  $H_0$  but instead  $\frac{3}{2}H_0$ . The estimate of  $H_0$  has thus been biased upward. It is  $\frac{3}{2}H_0$  that has the reported value of 70. The true value of  $H_0$  is thus only  $\frac{2}{3} \times 70 \approx 47$ . From the age formula  $t_0 = \frac{2}{3H_0}$ , the actual age of the Universe is about  $\frac{3}{2} \times 9bl \approx 14$  Glys in agreement with WMAP. The age problem disappears. (An otherwise accepted value of  $H_0 = 70$  yields an age of 9 bn years, an absurd result given that the oldest stars are known to be 12 bn years old.) The need to resort to a scenario of accelerating phase following a decelerating phase such as in [61] disappears.

<sup>19</sup> One way to interpret Formula (61) is that the beating rate of clocks in the regions the photons passed by slowed down in disproportion with the cosmic scale factor, viz.  $dt \propto a^{3/2}$  instead of  $a$  (due to the anisotropic exponent for time scaling  $\eta = \frac{3}{2}$ .) One should realize that in observational cosmology it is the shift in the photon frequency – instead of its wavelength – that is measured. Yet this technical detail is of secondary importance. Even if the redshift were measured in terms of the photon wavelength, the redshift formula is still:  $1 + z_{\text{observed}} = a^{-3/2}$ ; see Appendix G for explanation.

## 2. An alternative interpretation of Type Ia supernovae data:

Consider two supernovae  $A$  and  $B$  at distances  $3bn$  and  $6bn$  light years away from the Earth, viz.  $d_B = 2d_A$ . Standard cosmology dictates the redshift values of  $z_A$  and  $z_B \approx 2z_A$  for them (to first-order approximation.) However, light travelled faster in a distant past than it did in a more recent epoch. Thus, the  $B$ -photon covered twice as long the distance in less than twice the amount of time as compared with the  $A$ -photon. Having spent less time in transit than expected, the  $B$ -photon experienced less cosmic expansion than expected, and thus less redshift than standard cosmology predicts. Namely:  $z_B < 2z_A$ . This result means an upward slopping in the curve as we put the two supernovae's data on the  $d$  vs.  $z$  plot. Conversely, a supernova  $C$  with  $z_C = 2z_A$  must correspond to a distance greater than  $6bn$  light years, viz.  $d_C > d_B = 2d_A$ , and thus is a fainter object. This is precisely what being observed in the data of Type Ia supernovae [58, 59]. In Section 8 we shall show that the same value of  $H_0$  which resolves the age problem alluded above also gives the fit to the Type Ia supernovae data without any adjustable parameter whatsoever. No fudge factors, such as the amount of dark energy, are needed. The cosmic expansion is not accelerating; the observed discrepancy in Type Ia supernovae as compared to the critical expansion mode stems from the shortcoming of the standard cosmological formulation to take into account the effects of  $c$ -variation as function of the cosmic scale factor.

## 3. The horizon problem:

Observational data shows a highly uniform distribution (to the accuracy of  $10^{-5}$ ) of cosmic radiation across the horizon. This uniformity presents a serious challenge to the Friedmann cosmological model – the observed uniformity cannot reconcile with the fact that different segments of the current horizon are not causally connected. However, as we pointed out before, the Friedmann model does not take into account the variability of  $c$  as function of the cosmic scale factor  $a$ , per (58):  $c \propto a^{-1/2}$ , which means a much higher value of  $c$  in the baby universe and thus a larger cosmological horizon than what the Friedmann model dictates. Quantitatively, this scaling rule precisely produces an infinite value for the cosmological horizon (see Eq. (179)) as we shall explicitly show in Section 9.3. This result neatly explains the near uniformity in our current horizon since the entire universe was in causal contact before stretching out to its current state. The idea of employing a larger value of  $c$  in the past (about 60 orders of magnitude higher than now) to account for the horizon paradox was first advanced by Moffat [52] and subsequently by Albrecht and Magueijo [53]. What is new in our approach is an organic mechanism behind the variation of  $c$  via the role of the Ricci scalar as the scale setter per Postulate (II). Our resolution of the horizon paradox invokes no ad hoc assumptions as was done in the inflationary universe hypothesis.

## 4. The flatness problem:

Observational data further confirm that the universe is almost flat. This fact represents another serious challenge to the Friedmann model. Flatness is very difficult to achieve in the Friedmann model; it requires a fine tuning to an extraordinary level. Curvature-scaling gravity offers a natural explanation to the flatness problem as well. As the early universe expanded, the speed at which light travels decreased, according to the scaling rule (58):  $c \propto a^{-1/2}$ . The effective horizon thus shrank relatively to the universe's size. By the time of the decoupling event, at which point light began to travel freely, as the universe had grown several orders over in size and simultaneously the light speed had dropped several orders over in magnitude, the observable region within which light signals that emitted from could reach us today constituted a pocket several orders of magnitude smaller than the universe. This explains the flatness problem. In our resolution, we do not need to resort to any ad hoc assumptions as was done in the inflationary expansion hypothesis <sup>20</sup>.

Curvature-scaling gravity provides a natural, unified, coherent, comprehensive explanation – and parsimonious in the Occam Razor's sense – for the four most pressing problems in cosmology. Our explanation is model-independent (without resorting to any specific model of matter distribution in the universe); it is also derived from first principle, viz. Postulate (II) on the role of the Ricci scalar as the dynamical scale-setter. It is noteworthy that curvature-scaling gravity was not initially designed to resolve these problems but was motivated by our desire to understand the underlying agent that decides the size of things in our surroundings. In and of itself, curvature-scaling gravity is theory of gravity and spacetime, instead of a theory of cosmology alone.

If we subscribe to the new depiction of the spacetime manifold, a manifold which allows each local region to have its own length scale – the Ricci length – which has been in steady decline as the universe expands, then the Friedmann model misses out this very important property of spacetime – viz. the adaptation of  $c$  to the cosmic scale factor – and cannot be used as a theoretical foundation for cosmology. Such a problematic model would inevitably lead to theoretical

<sup>20</sup> Beside the flatness and horizon problems, the Friedmann model suffers Dicke's instability problem in which a tiny deviation away from the critical density would rapidly drive the universe away from its current state – this is also known as the oldness problem. Curvature-scaling gravity resolves this problem too. The evolution of the universe is governed by a generic scaling rule which strictly forces the universe to expand in the critical mode regardless of the its shape and/or its matter content and density. The robust evolution law prevents the universe from collapsing or expanding supercritically. The anthropic argument is not needed. See Section 9.5 for our explanation.

predictions irreconcilable with observational data. Foremost among its problems are the difficulties to account for Type Ia supernovae data, the age of the cosmos, the near uniform observable horizon, the near flatness of space, to name a few. The reason behind the difficulties for standard cosmology to account for observations is that the cosmos serves as the largest laboratory imaginable in which the effects of the Ricci scalar as a dynamical scale-setter strongly manifest themselves.

To reconcile its predictions with observations, standard cosmology has had to invent several ad hoc addenda to the Friedmann model: the cosmological constant (viz. the dark energy), the accelerated expansion, and the inflationary expansion, which introduce a new set of unsettling fine-tunings and difficulties. Given the unified resolution from curvature-scaling gravity for these once-troubling problems, these ad hoc supplementaries – which have nonetheless integrated into the vocabulary of cosmology – are unnecessary and lack a logical basis.

## 6 A nontrivial solution to the curvature-scaling field equation and its consequences in the physics of black holes

In this section, we shall derive a non-trivial solution to the field equations of curvature-scaling gravity in vacuo and explore its implications in the physics of black holes. The solution is found for the static spherically symmetric setup up to the first-order perturbative expansion in terms of the Mannheim-Kazanas parameter  $\gamma$  alluded in Section 4. The parameter  $\gamma$  was found in Mannheim's work [3–6] to be small for astronomical objects.

The solution presented below has a non-constant Ricci scalar which diverges on the event horizon. From the logical deductions regarding the variabilities in  $\hbar$  and  $c$  derived in the preceding section, the solution allows  $c$  to rise indefinitely and  $\hbar$  to drop to zero as one approaches the event horizon of a massive object in the de Sitter background. Right on the event horizon, as the Ricci scalar diverges logarithmically,  $c$  is infinite and  $\hbar$  vanishes. Quantum effects thus diminish on the event horizon. How this result affects the radiation of Schwarzschild-type black holes is an open, yet intriguing, issue which merits further investigations.

### The solution with non-constant Ricci scalar:

The full solution for the static spherically symmetric setup is given in Buchdahl's metric (23, 24) in Section 3. Generally, Buchdahl's solution admits spacetime configurations with non-constant Ricci scalar. In Section 4 we also found an exact solution (25, 29) which introduced a new term, the linear Mannheim-Kazanas  $\gamma r$  term in  $\Psi = \sqrt{1 - 3r_s\gamma - \frac{r_s}{r}} - \Lambda r^2 + \gamma r$  in the line element. This solution has a constant Ricci scalar, however. Our aim in this section is to find a solution with non-constant Ricci scalar.

We shall find a solution which is asymptotic to the constant- $\mathcal{R}$  solution (25, 29) which contains 3 parameters; the deviation from constant Ricci scalar will be specified by a new parameter, bringing the number of parameters to 4 as demanded by Buchdahl's treatment (see end of Section 3.) We shall express the metric in the form

$$ds^2 = e^\alpha \left[ -\Psi (dx^0)^2 + \frac{dr^2}{\Psi} + r^2 d\Omega^2 \right] \quad (62)$$

with two unknown functions  $\alpha$  and  $\Psi$ . Taking the cue from the solution (25, 29), we shall find the 2 functions  $\alpha$  and  $\Psi$  as perturbative expansion in terms of  $\gamma$ :

$$\begin{cases} \Psi &= \Psi_0 + \gamma \Psi_1 + \mathcal{O}(\gamma^2) \\ \alpha &= \gamma \Phi_1 + \mathcal{O}(\gamma^2) \end{cases} \quad (63)$$

with

$$\Psi_0 = 1 - \frac{r_s}{r} - \Lambda r^2. \quad (64)$$

We shall continue to call  $\gamma$  the Mannheim-Kazanas parameter, although their original utility of this parameter was limited to the context of dark matter (see Section 4.) Our derivation for  $\Psi_1$  and  $\Phi_1$  is detailed in Appendix D. Here is the outline of our derivation:

- For the two unknowns  $\Psi_1$  and  $\Phi_1$ , the two field equations needed are the  $tt$ - and the  $\theta\theta$ -components, which read

$$\left( \mathcal{R}_{tt} - \frac{1}{4} g_{tt} \mathcal{R} \right) \mathcal{R} = -\Gamma_{tt}^r \mathcal{R}' \quad (65)$$

$$\left( \mathcal{R}_{\theta\theta} - \frac{1}{4} g_{\theta\theta} \mathcal{R} \right) \mathcal{R} = -\Gamma_{\theta\theta}^r \mathcal{R}' \quad (66)$$

in which  $\mathcal{R}_{tt}$ ,  $\mathcal{R}_{\theta\theta}$  and  $\mathcal{R}$  are obviously involved  $\Psi_1$  and  $\Phi_1$ .

- Next, expand the two field equations in terms of  $\gamma$  order-by-order. The zero-order terms in  $\gamma$  disappear since we already specified  $\Psi_0 = 1 - \frac{r_s}{r} - \Lambda r^2$ . The first-order  $\gamma$  terms thus constitute two equations for  $\Psi_1$  and  $\Phi_1$ . We shall ignore terms with second- and higher-order in  $\gamma$ .
- The two equations for  $\Psi_1$  and  $\Phi_1$  just obtained (see Eqs. (312) and (313) in Appendix D) contain  $\Psi_1'''$  and  $\Phi_1'''$  with prime denoting derivative w.r.t. the radial coordinate  $r$ . Furthermore, these equations couple with each other. So, an outright elimination procedure would necessarily yield even higher derivatives and the problem thus looks intractable.



Despite this deceptive complexity, after a thicket of calculations detailed in Appendix D, we were able to gather complicated terms together<sup>21</sup> and obtain the formulae:

$$\Psi_1 = r - \frac{3}{2}r_s \quad (67)$$

$$\Phi_1 = -r - \epsilon \int \frac{dr}{r^2 \left(1 - \frac{r_s}{r} - \Lambda r^2\right)} \quad (68)$$

Despite the complexity of the equations, the expressions for  $\Psi_1$  and  $\Phi_1$  are remarkably neat. The metric up to first-order in  $\gamma$  is thus

$$\begin{aligned} ds^2 &= e^\alpha \left[ -\Psi(dx^0)^2 + \frac{dr^2}{\Psi} + r^2 d\Omega^2 \right] \\ \Psi &= 1 - \frac{r_s}{r} - \Lambda r^2 + \gamma \left( r - \frac{3}{2}r_s \right) + \mathcal{O}(\gamma^2) \\ \alpha &= -\gamma \left[ r + \epsilon \int \frac{dr}{r^2 \left(1 - \frac{r_s}{r} - \Lambda r^2\right)} \right] + \mathcal{O}(\gamma^2) \end{aligned} \quad (69)$$

and the Ricci scalar is

$$\mathcal{R} = 12\Lambda \left[ 1 - \gamma\epsilon \int \frac{dr}{r^2 \left(1 - \frac{r_s}{r} - \Lambda r^2\right)} \right] + \mathcal{O}(\gamma^2). \quad (70)$$

The metric now involves 4 parameters:

- $\Lambda$ , the de Sitter parameter, specifying the large-distance curvature.
- $r_s$ , the Schwarzschild radius (taken as a free parameter as a whole, instead of a combination of separate terms  $\frac{GM}{2c^2}$ ). It specifies the “strength” of the gravitational field source, as usual.
- $\gamma$ , the Mannheim-Kazanas parameter, specifying the linear potential term which Mannheim exploited in his theory of galactic rotation curves [3–6].
- An additional parameter  $\epsilon$ , allowing the Ricci scalar to deviate from constancy. We shall call it the anomalous curvature parameter.

The number of parameters is 4, precisely the number derived from Buchdahl’s general study of spherically symmetric systems. The last parameter,  $\epsilon$ , is not present in conformal gravity, e.g., in Mannheim-Kazanas’s solution. This parameter is specific to our solution since it appears in the phase factor which would have been conformally “gauged” away in conformal gravity. The anomalous curvature is only manifest if these three conditions are met:

- $\Lambda \neq 0$ , the metric must not be asymptotically flat.
- $\gamma \neq 0$ , the constant Mannheim-Kazanas centripetal acceleration must be in effect.
- $\epsilon \neq 0$ , the anomalous parameter, self-evidently, must not vanish.

The parameter  $\epsilon$  plays a central role in our prediction below, in which we predict new behaviors for the Schwarzschild-type black holes. The four parameters are dimensional:  $[\Lambda] = [\text{length}]^{-2}$ ,  $[r_s] = [\text{length}]$ ,  $[\gamma] = [\text{length}]^{-1}$ ,  $[\epsilon] = [\text{length}]^2$ , all helping set the scale for the black holes. They depend on the boundary conditions.

In the limit of small  $\gamma$  and  $\epsilon = 0$ , solution (69) is compatible with the constant- $\mathcal{R}$  solution (25, 29) in Section 4. To see this, let us Taylor expand (30) with respect to  $\gamma$  of the solution in Section 4:

$$\begin{cases} \gamma & \triangleq -\frac{\kappa}{3r_s} \\ \Psi & = \sqrt{1 - 3r_s\gamma - \frac{r_s}{r} - \Lambda r^2} + \gamma r \\ e^\alpha & = \left(1 + \frac{1 - \sqrt{1 - 3r_s\gamma}}{3r_s} r\right)^{-2} \end{cases} \quad (71)$$

to obtain

$$\begin{cases} \Psi & = 1 - \frac{r_s}{r} - \Lambda r^2 + \gamma \left( r - \frac{3}{2}r_s \right) + \mathcal{O}(\gamma^2) \\ \alpha & = -2 \ln \left( 1 + \frac{1 - \sqrt{1 - 3r_s\gamma}}{3r_s} r \right) = -\gamma r + \mathcal{O}(\gamma^2). \end{cases} \quad (72)$$

These Taylor series are compatible with solution (69) with  $\epsilon = 0$ .

<sup>21</sup> We first tried deriving the solution in the form of Laurent series w.r.t.  $r$ . The series we found appeared to possess a number of interesting properties, which guided us to reverse-engineer the solution. The final outcome is the analytical formulae for  $\Psi_1$  and  $\Phi_1$  as in (67, 68).

### Prediction – new properties of Schwarzschild-type black holes:

Solution (69) is interesting in either limit  $r \rightarrow 0$  or  $r \rightarrow r_s$ . We shall focus on the limit of  $r \rightarrow r_s$  in what follows. Let us consider a small  $\Lambda$  and denote  $r_*$  to be the root of the algebraic equation  $1 - \frac{r_s}{r} - \Lambda r^2 = 0$  that is closest to  $r_s$ . When  $\epsilon = 0$ , the event horizon is  $r_*$  since  $\Psi_0 \rightarrow 0$  as  $r \rightarrow r_*$  obviously.

When  $\epsilon \neq 0$  and as  $r \rightarrow r_*$ , the Ricci scalar (70) takes the form

$$\mathcal{R} \simeq \frac{\Lambda \gamma \epsilon}{r_*} \ln \left| 1 - \frac{r_*}{r} \right|. \quad (73)$$

The Ricci scalar diverges logarithmically as one approaches the event horizon. This leads to an intriguing phenomenon. As the observer free falls into a very massive object, the ruler and all objects in his surroundings would shrink, per  $dx \propto a_{\mathcal{R}} = |\mathcal{R}|^{-1/2}$ , and his wristwatch speeds up disproportionately, per  $dt \propto a_{\mathcal{R}}^{3/2} = |\mathcal{R}|^{-3/4}$ . Note that the effect of time flow speeding up is an opposite effect to the standard gravitational redshift which is also in play <sup>22</sup>. As the result of his clock and ruler scaling, the observer free falling in the black hole would experience an increasing value of  $c$  (once again, superluminality is strictly forbidden) and a decreasing value of  $\hbar$ , according to

$$c \simeq \left| \ln \left| 1 - \frac{r_*}{r} \right| \right|^{1/4} \quad (74)$$

$$\hbar \simeq \left| \ln \left| 1 - \frac{r_*}{r} \right| \right|^{-1/4} \quad (75)$$

Right on the event horizon,  $\hbar = 0$  and  $c \rightarrow \infty$ . The variations are only logarithmical. Nonetheless, whether and/or how the diminishment of quantum effects alter the behavior of black holes such as their radiative properties, quantitatively and/or qualitatively, deserve a thorough examination in future research.

We thus predict the novel properties of Schwarzschild-type black holes residing on a de Sitter background. Obviously the tests of our prediction regarding the behavior of black holes lie beyond the reach of man's current technology. Our initial enquiry in the arena of black holes is thus purely a theoretical adventure, which could help elucidate the structure of spacetime in the strong gravitational field limit. In addition, our mathematical endeavors in deriving the solutions (25, 29) and (69, 70) illustrate the tractability of curvature-scaling gravity.

<sup>22</sup> We do not, however, expect the correction to the standard gravitational redshift to be material for the solar system because we do not expect the Ricci scalar to vary to any substantial amount within the solar system.

## 7 Coupling of curvature-scaling gravity with matter: a departure from conventional construction of Lagrangian

Let us define three parameters with their units indicated respectively:

$$\hat{h} \triangleq \frac{\hbar}{\sqrt{a_{\mathcal{R}}}} \sim \frac{[\text{mass}] \cdot [\text{length}]^{3/2}}{[\text{time}]} \quad (76)$$

$$\hat{c} \triangleq c \sqrt{a_{\mathcal{R}}} \sim \frac{[\text{length}]^{3/2}}{[\text{time}]} \quad (77)$$

$$\zeta \triangleq \frac{\hat{h}}{\hat{c}} \sim [\text{mass}] \quad (78)$$

with  $a_{\mathcal{R}} = |\mathcal{R}|^{-1/2}$  being the Ricci length. Given the scaling rules (57) and (58), each of these parameters has a fixed value at all points on the manifold. Conversely, the Planck constant and speed of light are dependent on the Ricci scalar, per

$$\hbar = \hat{h} a_{\mathcal{R}}^{1/2} \quad (79)$$

$$c = \hat{c} a_{\mathcal{R}}^{-1/2} \quad (80)$$

Note that beside  $\zeta$  which is independent of  $\mathcal{R}$ , the dimensionless fine-structure constant  $\alpha \triangleq e^2 / (\hbar c) = e^2 / (\hat{h} \hat{c})$  is also independent of  $\mathcal{R}$ .

The standard algorithm of covariantising an action (also known as “the principle of minimal coupling”) consists of the following steps:

- Write down the Lorentz-invariant action in special relativity.
- Replace the Minkowski metric  $\eta_{\mu\nu}$  by  $g_{\mu\nu}$  whenever applicable.
- Replace a partial derivative  $\partial_\mu$  by the covariant derivative  $\nabla_\mu$  whenever applicable.
- Replace the volume element  $d^4x$  by its (invariant) counterpart  $d^4x \sqrt{g}$ .

By construction, these equations are tensorial and true in the absence of gravity and hence satisfy the general covariance principle. What needs to be done in curvature-scaling gravity are three additional steps:

- When writing down the Lorentz-invariant action in special relativity, cast it in the terms of dimensionless quantities. Then proceed to replace  $\eta_{\mu\nu} \rightarrow g_{\mu\nu}$ ,  $\partial_\mu \rightarrow \nabla_\mu$ ,  $d^4x \rightarrow d^4x \sqrt{g}$  as previously required.
- Replace  $\hbar$  by  $\hat{h}$  and  $c$  by  $\hat{c}$  whenever applicable.
- Finally, make two following replacements, whenever applicable:

$$\begin{cases} dx^\mu \rightarrow d\tilde{x}^\mu = \frac{dx^\mu}{a_{\mathcal{R}}} = |\mathcal{R}|^{1/2} dx^\mu \\ \nabla_\mu \rightarrow \tilde{\nabla}_\mu = a_{\mathcal{R}} \nabla_\mu = |\mathcal{R}|^{-1/2} \nabla_\mu \end{cases} \quad (81)$$

In particular, the volume element is replaced as

$$d^4x \sqrt{g} \rightarrow d^4x \mathcal{R}^2 \sqrt{g} \quad (82)$$

Let us start from the action of the massive scalar field theory

$$\mathcal{S} \simeq \int d^4x \left[ -\frac{1}{2} \hbar^2 c^2 \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 c^4 \phi^2 \right]. \quad (83)$$

We deliberately cast in terms of dimensionless quantities

$$\mathcal{S} \simeq \int d^4x \left[ -\frac{1}{2} \left( \frac{\hat{h}}{mc} \partial^\mu \phi \right) \left( \frac{\hat{h}}{mc} \partial_\mu \phi \right) - \frac{1}{2} \phi^2 \right]. \quad (84)$$

Upon covariantising it, we get (for scalar fields,  $\nabla_\mu = \partial_\mu$ , though)

$$\mathcal{S} \simeq \int d^4x \sqrt{g} \left[ -\frac{1}{2} \left( \frac{\hat{h}}{mc} \nabla^\mu \phi \right) \left( \frac{\hat{h}}{mc} \nabla_\mu \phi \right) - \frac{1}{2} \phi^2 \right]. \quad (85)$$

Making the replacements  $\hbar \rightarrow \hat{\hbar}$ ,  $c \rightarrow \hat{c}$  and (81), we obtain the action

$$\mathcal{S}_{CSG} = \int d^4x \mathcal{R}^2 \sqrt{g} \left[ -\frac{1}{2} \left( \frac{\hat{\hbar}}{m\hat{c}} |\mathcal{R}|^{-1/2} \nabla^\mu \phi \right) \left( \frac{\hat{\hbar}}{m\hat{c}} |\mathcal{R}|^{-1/2} \nabla_\mu \phi \right) - \frac{1}{2} \phi^2 \right] \quad (86)$$

with CSG abbreviating curvature-scaling gravity. In terms of  $\zeta$ , it is recast as

$$\mathcal{S}_{CSG} = \int d^4x \mathcal{R}^2 \sqrt{g} \left[ -\frac{\zeta^2}{2m^2 |\mathcal{R}|} \nabla^\mu \phi \nabla_\mu \phi - \frac{1}{2} \phi^2 \right]. \quad (87)$$

The right-hand-side is dimensionless. More generally, for a covariantized action cast in dimensionless quantities

$$\mathcal{S} \simeq \int d^4x \sqrt{g} \mathcal{L}_m(\phi, \nabla_\mu \phi), \quad (88)$$

the curvature-scaling gravity action is

$$\mathcal{S}_{CSG} = \int d^4x \mathcal{R}^2 \sqrt{g} \mathcal{L}_m(\phi, |\mathcal{R}|^{-1/2} \nabla_\mu \phi) \quad (89)$$

This action contains all we need: the matter field  $\phi$  and the gravitational field  $g_{\mu\nu}$ . Functionally varying  $\phi$  while holding  $g_{\mu\nu}$  fixed produces the equation of motion for the matter field. Functionally varying  $g_{\mu\nu}$  while holding  $\phi$  fixed produces the gravitational field equations. As can be seen from (89), gravity – i.e., the metric tensor  $g_{\mu\nu}$  – organically arises from matter fields via five simultaneous routes:

- $|\mathcal{R}|^{-1/2} \nabla_\mu$  in the Lagrangian of matter  $\mathcal{L}_m$ ;
- $\mathcal{R}^2 d^4x$  of the volume element;
- the Jacobian  $\sqrt{g}$ , as usual;
- the covariant derivatives  $\nabla_\mu$ ;
- the contravariant derivatives  $\nabla^\mu = g^{\mu\nu} \nabla_\nu$ .

Each of these terms participates in the Lagrangian for a legitimate reason, rather than an ad hoc formality.  $\mathcal{S}_{CSG}$  explicitly excludes the cosmological term as well as the Einstein-Hilbert term  $\mathcal{R}$ .

We stress that, in our approach, there is no separate Lagrangian for a “free” gravitational field. Gravitational field always couples with matter; it does not exist in isolation. Our approach is thus a radical departure from the standard practice. Traditionally, one would first write down the Lagrangian for a “free” gravitational field called  $\mathcal{L}_{free}$ ; after that, one would augment it with the Lagrangian for the matter part  $\mathcal{L}_m$  which is often let couple with the gravitational field via the minimal coupling. There also needed be a factor – viz., the coupling constant – to specify the relative weight for  $\mathcal{L}_m$  against  $\mathcal{L}_{free}$ . For example, the Einstein-Hilbert full action is ( $c = 1$ )

$$\mathcal{S}_{EH} = -\frac{1}{16\pi G} \int d^4x \sqrt{g} \mathcal{R} + \int d^4x \sqrt{g} \mathcal{L}_m, \quad (90)$$

the Weyl action of conformal gravity [3] is

$$\mathcal{S}_W = -\alpha_g \int d^4x \sqrt{g} \left[ \mathcal{R}_{\mu\nu\lambda\sigma} \mathcal{R}^{\mu\nu\lambda\sigma} - 2\mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} + \frac{1}{3} \mathcal{R}^2 \right] + \int d^4x \sqrt{g} \mathcal{L}_m, \quad (91)$$

and the quadratic gravity (QG) action is

$$\mathcal{S}_{QG} = -\alpha \int d^4x \sqrt{g} \mathcal{R}^2 + \int d^4x \sqrt{g} \mathcal{L}_m \quad (92)$$

All gravitational theories devoted their focus on the form of the  $\mathcal{L}_{free}$  piece of the “free” gravitational field. We take a radically different route: in our approach, it is the matter part that receives the primary focus. There is no “free” gravitational field in our approach; gravity does not exist in isolation. The single term in action (89) contains both the matter field and the gravitational field simultaneously.

We need to address two issues:

- Since there is no  $\mathcal{L}_{free}$  for the “free” gravitational field, how would the Lagrangian of gravitation field in vacuo be defined? We formally define the vacuo as the configuration in which matter has no dynamics – i.e., the derivatives of the matter field vanish and the density of matter field is subsequently set equal zero. This definition is justified if we interpret the uniform zero-point energy background as vacuum. In so doing, action (87) yields

$$\mathcal{S}_{vacuo} = \int d^4x \sqrt{g} \mathcal{R}^2 \quad (93)$$

which coincides with action (15) considered in Section (3). Action (82) happens to imitate the quadratic action (92) in vacuo. Note however that, in the presence of matter, our action (89) is not the full quadratic action (92). In addition, our action (89) was inspired from Postulate (II) in Section 1 which mandates that only dimensionless quantities be of relevance for physical laws.

- The Newton constant, viz. the gravitational constant  $G$ , is absent in our action. There is no input parameter that specifies the strength of coupling between matter and the gravitation field. This is a rather surprising aspect of our approach. The effective coupling can arise when we compare the Lagrangian at two different locations relative to each other, that is to say, the relative ratio of matter dynamics and/or density at the two locations. The vacuum corresponds to a region in which the dynamics of matter, viz.  $\partial^\mu \phi$ , is negligible; in such a situation, the Lagrangian of vacuo reads:  $\mathcal{S}_{vacuo} = \int d^4x \sqrt{g} \mathcal{R}^2 \rho$ . On the other hand, in the region where the dynamics and/or density of matter are not negligible, gravity couples with matter in full form:  $\mathcal{S}_{CSG} = \int d^4x \sqrt{g} \mathcal{R}^2 \mathcal{L}_m \left( |\mathcal{R}|^{-1/2} \partial_\mu \phi \right)$ . The “strength” of the coupling of matter  $\mathcal{L}_m$  to gravity is seen as a result of a relative comparison of matter’s density and/or dynamics in different regions.<sup>23</sup>

### The field equations of the metric components:

Upon functionally varying  $g_{\mu\nu}$  while keeping the matter field fixed in the action of curvature-scaling gravity

$$\mathcal{S}_{CSG} = \int d^4x \sqrt{g} \mathcal{R}^2 \mathcal{L}_m \left( |\mathcal{R}|^{-1/2} \nabla_\mu \right), \quad (94)$$

we obtain the field equations

$$(\mathcal{R}_{\mu\nu} - \nabla_\mu \nabla_\nu + g_{\mu\nu} \square) \left[ 2\mathcal{R} \mathcal{L}_m + \mathcal{R}^2 \frac{\partial \mathcal{L}_m}{\partial \mathcal{R}} \right] = \frac{1}{2} \mathcal{R}^2 T_{\mu\nu}, \quad (95)$$

or, equivalently

$$(\mathcal{R}_{\mu\nu} - \nabla_\mu \nabla_\nu + g_{\mu\nu} \square) \left[ \mathcal{R} \left( \mathcal{L}_m + \frac{1}{2} |\mathcal{R}| \frac{\partial \mathcal{L}_m}{\partial |\mathcal{R}|} \right) \right] = \frac{1}{4} \mathcal{R}^2 T_{\mu\nu}. \quad (96)$$

The stress-energy tensor is, as usual

$$T_{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\partial (\sqrt{g} \mathcal{L}_m)}{\partial g^{\mu\nu}} = g_{\mu\nu} \mathcal{L}_m - 2 \frac{\partial \mathcal{L}_m}{\partial g^{\mu\nu}} \quad (97)$$

but  $\mathcal{L}_m$  now contains an extra dependence on  $\mathcal{R}$  via  $|\mathcal{R}|^{-1/2} \nabla_\mu \phi$ . Note that the matter Lagrangian  $\mathcal{L}_m$  participates on both sides of the field equations (96). The coupling parameter (viz. the Newton constant  $G$ ) is absent from the field equations.

In vacuum,  $\mathcal{L}_m$  does not contain any derivatives  $\partial^\mu \phi$ ; as such, it does not contain  $\mathcal{R}$ . The field equations (96) recovers the field equation in vacuo (compared with Eq. (18)):

$$(\mathcal{R}_{\mu\nu} - \nabla_\mu \nabla_\nu + g_{\mu\nu} \square) \mathcal{R} = \frac{1}{4} \mathcal{R}^2 g_{\mu\nu},$$

or, upon taking the trace

$$\left( \mathcal{R}_{\mu\nu} - \frac{1}{4} g_{\mu\nu} \mathcal{R} \right) \mathcal{R} = \nabla_\mu \nabla_\nu \mathcal{R}.$$

<sup>23</sup> The Newton constant can emerge in an approximation outlined below. Set the vacuo density to be a small value  $\rho$  and take the scalar curvature to vary slowly around  $\mathcal{R}_0$ . Next, approximate  $\mathcal{R}$  in  $\mathcal{L}_m$  by  $\mathcal{R}_0$ , and linearize  $\mathcal{R}^2$  around  $\mathcal{R}_0$ :  $\mathcal{R}^2 \approx 2\mathcal{R}_0 \mathcal{R} - \mathcal{R}_0^2$ . Then approximate

$$\mathcal{R}^2 \mathcal{L}_m = \mathcal{R}^2 \rho \left[ 1 + \frac{1}{\rho} (\mathcal{L}_m - \rho) \right] \approx \rho \left[ \mathcal{R}^2 + \frac{\mathcal{R}_0^2}{\rho} \tilde{\mathcal{L}}_m \right] \approx 2\mathcal{R}_0 \rho \left[ \mathcal{R} - \frac{1}{2} \mathcal{R}_0 + \left( \frac{1}{2\rho} \mathcal{R}_0 \right) \tilde{\mathcal{L}}_m \right]$$

in which  $\tilde{\mathcal{L}}_m \triangleq \mathcal{L}_m - \rho$  is the remaining matter part after the vacuum density is subtracted out. The terms in the last square bracket mimic the Einstein-Hilbert action with coupling  $G = \frac{c^4}{16\pi\rho} \mathcal{R}_0$  and a cosmological term  $\Lambda = -\frac{1}{2} \mathcal{R}_0$ . These manipulations are only meant to illustrate how the Newton constant  $G$  could emerge in an approximation to our Lagrangian in the configuration of slow varying field.

**The divergence of stress-energy tensor:**

Taking the divergence of the field equation of curvature-scaling gravity (96), we get <sup>24</sup>

$$\nabla^\mu (\mathcal{R}^2 T_{\mu\nu}) = \left( \mathcal{L}_m + \frac{1}{2} |\mathcal{R}| \frac{\partial \mathcal{L}_m}{\partial |\mathcal{R}|} \right) \nabla_\nu \mathcal{R}^2 \quad (98)$$

or, equivalently

$$\mathcal{R}^2 \nabla^\mu T_{\mu\nu} = \nabla^\mu \mathcal{R}^2 \left( g_{\mu\nu} \mathcal{L}_m + \frac{1}{2} g_{\mu\nu} |\mathcal{R}| \frac{\partial \mathcal{L}_m}{\partial |\mathcal{R}|} - T_{\mu\nu} \right) = 2 \nabla^\mu \mathcal{R}^2 \left( \frac{\partial \mathcal{L}_m}{\partial g^{\mu\nu}} + \frac{1}{2} g_{\mu\nu} |\mathcal{R}| \frac{\partial \mathcal{L}_m}{\partial |\mathcal{R}|} \right).$$

Finally, we arrive at

$$\nabla^\mu T_{\mu\nu} = 2 \left( \frac{\partial \mathcal{L}_m}{\partial g^{\mu\nu}} + \frac{1}{2} g_{\mu\nu} |\mathcal{R}| \frac{\partial \mathcal{L}_m}{\partial |\mathcal{R}|} \right) \nabla^\mu \ln \mathcal{R}^2. \quad (99)$$

By virtue of (99), in curvature-scaling gravity, energy and momentum are no longer conserved. Energy and momentum are measured in each local spacetime pocket and are dependent on the Ricci scalar. <sup>25</sup>

<sup>24</sup> We will need to utilize  $\nabla^\mu \mathcal{R}_{\mu\nu} = \frac{1}{2} \nabla_\nu \mathcal{R}$  and  $\nabla^\mu \nabla_\mu \nabla_\nu f - \nabla_\nu \nabla^\mu \nabla_\mu f = \mathcal{R}_{\mu\nu} \nabla^\mu f$  which holds for any scalar function  $f$ .

<sup>25</sup> As is discussed in Appendix A, energy of every system (be it electromagnetic, or a relativistic energy level of the Hydrogen atom, or a massive object) scales as  $a_{\mathcal{R}}^{-1}$  (whereas momentum, on the other hand, scales as  $a_{\mathcal{R}}^{-1/2}$ .) This effect of “energy loss” has already been known for radiation in the expanding universe via the “Doppler theft”. Curiously, standard cosmology exempts other forms of matter, such as the baryonic matter, from the energy loss effect. This treatment is unwarranted for two reasons:

- The binding energy in atoms is also of electromagnetic nature. It then cannot avoid the “Doppler theft”. The fact that galaxies resists the cosmic expansion is irrelevant.
- Radiation and matter were treated differently in standard cosmology. This is in violation of Postulate (I) which posits that physical laws retain their form in every local spacetime pocket. As the universe expands, both radiation and matter must transform exactly the same; i.e., they both have to experience the “Doppler theft”.

As we shall discuss in Section 9, this practice in standard cosmology arose from the problematic Friedmann model which does not take into account the variation of  $c$  as function of the cosmic scale.

## 8 Implications of curvature-scaling gravity in cosmography

Per the equivalence principle, physical laws are valid locally in every pocket of spacetime; namely, physical laws retain their form in all local regions on the manifold. Curvature-scaling gravity extends the equivalence principle further by requiring that the parameters specifying the physical laws are also valid only locally. The Ricci scalar  $\mathcal{R}$  serves as the local scale setter for all physical processes that take place in any given local region. As a result, the observer's clock rate scales anisotropically per (56),  $dt \propto a_{\mathcal{R}}^{3/2}$  (with the Ricci length being defined as  $a_{\mathcal{R}} \triangleq |\mathcal{R}|^{-1/2}$ ), and the Planck constant and the speed of light are dependent on the Ricci scalar per (57),  $\hbar \propto |\mathcal{R}|^{-1/4}$ , and (58),  $c \propto |\mathcal{R}|^{1/4}$ . We emphasize that – in contrary to long-standing belief – the variability of  $c$  does not violate causality or any principle of relativity – the Michelson-Morley finding, the Lorentz symmetry, the relativity principle, the equivalence principle, and the general covariance principle. The Michelson-Morley experiment only establishes the equality of light speed at one location regardless of the light beam's direction. It does not enforce the equality of light speed at different locations or at different time points. (See Section 5 for our elaboration.)

The impacts of the  $c$ -variability are most manifest in cosmology. As the universe expands, its Ricci scalar drops, leading to a progressive decline in the value of light speed since  $c \propto |\mathcal{R}|^{1/4}$ . Over billions years, the accumulation of  $c$ -variation shows up in observational data. A proper treatment of cosmology must then take into account the effects of  $c$ -variation as the universe expands. Failure to do so leads to difficulties in explaining observational data (unless one augments theoretical cosmology with ad hoc assumptions and fudge agents, such as dark energy, accelerated expansion, and inflationary expansion.)

Standard cosmology neglects to take into account the dependence of light speed on the Ricci scalar (i.e., the cosmic scale factor). As a result, it produced a series of flawed redshift formulae based on which observational data were analyzed. In this section, we shall revise these redshift formulae by incorporating the  $c$ -variability. The outline of this section is:

- We first include the  $c$ -variation in the Robertson-Walker metric, resulting in the modified Robertson-Walker metric.
- We then revise the Lemaitre redshift formula. The exponent  $\frac{3}{2}$  in the time anisotropy  $dt \propto a_{\mathcal{R}}^{3/2}$  enters the exponent of the scale factor in the modified Lemaitre redshift formula. Due to this  $\frac{3}{2}$  factor (which is missing in the standard redshift formula), the Hubble law needs be revised and the Hubble constant reduced. This is the key to resolving the age problem in cosmology.
- We next modify the distance-redshift relationship and the luminosity-redshift relationship. Formula (156) is the centerpiece for our reassessment of Type Ia supernovae data obtained in [58–60].
- We offer an alternative interpretation of the Type Ia supernovae data in [58–60]. Our interpretation is based on the variation of  $c$  as function of the cosmic scale factor, bypassing the standard explanation of an accelerated expansion.

For the sake of comparison, the derivations below are presented both in the traditional framework and in our approach. Correspondingly, the calculations are based on the traditional RW metric and the modified RW metric.

### 8.1 The modified Robert-Walker metric

The Robertson-Walker (RW hereafter) metric starts with the assumption of homogeneity and isotropy of space. It also assumes that the spatial component of the metric can be time-dependent. All of the time dependence is in the function  $a(t)$ , known as the “scale factor”. The RW metric is the only one that is spatially homogeneous and isotropic [46–49]. This is a geometrical result and is not tied to the equations of gravitation field.

The RW metric has been determined to be

$$\begin{aligned} ds^2 &= c_0^2 dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right] \\ d\Omega^2 &= d\theta^2 + \sin^2 \theta d\phi^2 \end{aligned} \quad (100)$$

where  $a(t)$  is the global scale factor of the universe and is a function of the cosmic time  $t$  only (with  $a(t_0) = 1$  for at our current time  $t_0$ ),  $k$  the curvature determining the shape of the universe (open/flat/closed for  $k > 0$ ,  $k = 0$ ,  $k < 0$  respectively.) Also recall that the RW metric assumes a constant speed of light which we denoted as  $c_0$  in (100).

When applying curvature-scaling gravity to cosmology, we retain the homogeneity and isotropy of space. As such, the RW metric remains applicable with the only modification in the dependence of  $c$  on the Ricci scalar (viz. the cosmic scale factor <sup>26</sup>) By virtue of the scaling rule (58),  $c = c_0 a^{-1/2}$ , we arrive at the modified Robertson-Walker metric:

$$ds^2 = \frac{c_0^2}{a(t)} dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right] \quad (101)$$

<sup>26</sup> We shall prove in Section 9.1 that the Ricci length  $a_{\mathcal{R}}$  is indeed proportional to the scale factor  $a$ . Therefore, we use  $a$  in place of  $a_{\mathcal{R}}$  in this current section.

in which  $c_0$  is the speed of light measured in the outer space (subject to cosmic expansion) at our current time.

## 8.2 Modification to Lemaitre's redshift formula

### 8.2.1 For the traditional RW metric:

Light wave travels in the null geodesics,  $ds^2 = 0$ . The traditional RW metric (100) can be recast as

$$ds^2 = a^2(t) \left[ \frac{c_0^2}{a^2(t)} dt^2 - \frac{dr^2}{1 - kr^2} - r^2 d\Omega^2 \right]. \quad (102)$$

The null geodesics for a light wave traveling from a galaxy toward Earth (viz.  $d\Omega = 0$ ) is thus

$$\frac{c_0 dt}{a(t)} = \frac{dr}{\sqrt{1 - kr^2}}. \quad (103)$$

Denote  $t_e$  and  $t_o$  the emission and observation timepoints of the light wave, and  $r_e$  the comoving distance of the galaxy from Earth. From (103) we have

$$\int_{t_e}^{t_o} \frac{c dt}{a(t)} = \int_{r_e}^0 \frac{dr}{\sqrt{1 - kr^2}}. \quad (104)$$

The next wavecrest to leave the galaxy at  $t_e + \delta t_e$  and arrives at Earth at  $t_o + \delta t_o$  satisfies

$$\int_{t_e + \delta t_e}^{t_o + \delta t_o} \frac{c dt}{a(t)} = \int_{r_e}^0 \frac{dr}{\sqrt{1 - kr^2}}. \quad (105)$$

Subtracting the two equations yields

$$\frac{\delta t_o}{a(t_o)} = \frac{\delta t_e}{a(t_e)}. \quad (106)$$

It is important to realize that in observational astronomy it is the shift in frequency – instead of wavelength – in the light spectrum that is used to determine the redshift of the distant light source. In terms of frequency shift, the redshift is defined as

$$z \triangleq \frac{\nu_e - \nu_o}{\nu_o}. \quad (107)$$

In standard cosmology, since  $c$  is a constant, the conversion from frequency to wavelength ( $\lambda = c/\nu$ ) is trivial, yielding an equivalent formula

$$z = \frac{\lambda_o - \lambda_e}{\lambda_e}. \quad (108)$$

By virtue of (106), the observed frequency is related to the emitted frequency by (since  $\nu = 1/\delta t$ ):

$$\frac{\nu_o}{\nu_e} = \frac{\delta t_e}{\delta t_o} = \frac{a(t_e)}{a(t_o)}. \quad (109)$$

We thus have

$$z = \frac{a(t_o)}{a(t_e)} - 1. \quad (110)$$

With  $t_o = t_0$  and  $a(t_o) = a(t_0) = 1$ , the traditional Lemaitre redshift formula is thus

$$1 + z = a^{-1}(t_e). \quad (111)$$

### 8.2.2 For the modified RW metric:

The modified RW metric (101) can be recast as

$$ds^2 = a^2(t) \left[ \frac{c_0^2}{a^3(t)} dt^2 - \frac{dr^2}{1 - kr^2} - r^2 d\Omega^2 \right]. \quad (112)$$

The null geodesics for a light wave traveling radially is thus

$$\frac{c_0 dt}{a^{3/2}(t)} = \frac{dr}{\sqrt{1 - kr^2}}, \quad (113)$$



leading to

$$\int_{t_e}^{t_o} \frac{c_0 dt}{a^{3/2}(t)} = \int_{r_e}^0 \frac{dr}{\sqrt{1 - kr^2}} \quad (114)$$

and

$$\int_{t_e + \delta t_e}^{t_o + \delta t_o} \frac{c_0 dt}{a^{3/2}(t)} = \int_{r_e}^0 \frac{dr}{\sqrt{1 - kr^2}}. \quad (115)$$

Subtracting the two equations yields:

$$\frac{\delta t_o}{a^{3/2}(t_o)} = \frac{\delta t_e}{a^{3/2}(t_e)} \quad (116)$$

from which, the observed frequency is related to the emitted frequency by

$$\frac{\nu_o}{\nu_e} = \frac{\delta t_e}{\delta t_o} = \frac{a^{3/2}(t_e)}{a^{3/2}(t_o)}. \quad (117)$$

The redshift parameter defined in (107),  $z = \frac{\nu_e - \nu_o}{\nu_o}$ , satisfies the modified Lemaitre redshift formula

$$1 + z = a^{-3/2}(t_e). \quad (118)$$

The most striking feature of (118) is the appearance of the time anisotropy exponent  $\eta = \frac{3}{2}$  in the scale factor<sup>27</sup>. This feature drastically alters the Hubble law and the estimation of the Hubble constant, which we shall show momentarily.

### 8.3 The modified Hubble law and the corrected value for Hubble constant

From the definition of the Hubble constant

$$H_0 \triangleq \left. \frac{\dot{a}}{a} \right|_{t=t_0} \quad (119)$$

and for a small time difference  $t_o - t_e$ , we get

$$a(t_e) = 1 - H_0(t_0 - t_e) + \dots \quad (120)$$

Denote  $d = c_0(t_0 - t_e)$  as the distance from the Earth to a galaxy. For small  $z$  and  $d$ , the Taylor expansion of the modified Lemaitre redshift formula (118) combined with (120) yields

$$\begin{aligned} 1 + z &= a^{-3/2}(t_e) \\ &= (1 - H_0(t_0 - t_e) + \dots)^{-3/2} \\ &= 1 + \frac{3}{2}H_0(t_0 - t_e) + \dots \\ &= 1 + \frac{3}{2}H_0 \frac{d}{c_0} + \dots \end{aligned} \quad (121)$$

We thus obtain the modified Hubble law:

$$z = \frac{3}{2}H_0 \frac{d}{c_0}. \quad (122)$$

Compared with the conventional Hubble law (which can be obtained from the Taylor expansion of the traditional Lemaitre redshift formula (111)), where the speed of light is explicitly restored:

$$z = H_0 \frac{d}{c_0}, \quad (123)$$

the modified Hubble law (122) acquires an additional prefactor of  $\frac{3}{2}$ . That means the redshift  $z$  detected in the astronomer's telescope was enhanced by a factor of  $\frac{3}{2}$ , the exponent in the anisotropic time scaling per (56),  $dt \propto a_{\mathcal{R}}^{3/2}$ . As such, to back out the actual scale factor at the distance galaxy, the astronomer has to reduce his reading of  $z$  by a factor of  $\frac{3}{2}$ . Without doing this, she would unknowingly overestimate the Hubble constant. This overestimation is precisely what happened in standard cosmology.

To put it another way, in the plot of  $z$  versus  $\frac{d}{c_0}$ , it is the  $\frac{3}{2}H_0$  what is the slope of the line. If the astronomer mistook the slope to be  $H_0$ , he would inadvertently overestimate the Hubble constant. With the reported value for  $H_0$  being 71, in the light of our reasoning, the actual value for  $H_0$  should – per Eq. (122) – be only  $\frac{2}{3} \times 71 \approx 47$ .

<sup>27</sup> As we noted before, in observational cosmology it is the shift in frequency – instead of wavelength – that is used to determine the redshift. This detail is often overlooked in theoretical treatments. Notwithstanding this technical detail however, we shall show in Appendix G that as long as the galaxy that emitted the photons and the Earth-based telescope are immune from cosmic expansion, the shift in the photon wavelength, defined as in (108), remains precisely the modified Lemaitre redshift formula (118) in which  $a(t_e)$  is the cosmic scale factor at the outskirts of the galaxy that emitted the photons and  $a(t_o)$  is the cosmic scale factor at the outskirts of our Milky Way. Namely, the prefactor of  $3/2$  still participates.

The reduced Hubble constant has two remarkable consequences:

- It increases the age of the universe by a factor of  $\frac{3}{2}$  to about 13.8 Gly. This is the key to resolving the age problem bypassing the need for an accelerating phase following a deceleration phase as advocated in [61].
- It leads to a lower critical density (since  $\rho_c \propto H_0^2$ ) than previous thought by a factor of  $\left(\frac{3}{2}\right)^2$ . This is the key to resolve the budgetary shortfall problem without the need of dark energy.

The reduction of  $\frac{3}{2}$  for the Hubble constant is not the end of the story yet, however. We shall shortly show that there are two additional sources of correction: (i) The first is a new distance-redshift formula in our model as compared to that in the  $\Lambda$ CDM model; (ii) The other is an extra modification factor when one converts the luminosity distance into the proper distance, a practice of importance for high- $z$  objects, such as Type Ia supernovae.

## 8.4 The modified distance-redshift relationship

### 8.4.1 For the Lambda-CDM model:

Let us first re-derive the distance-redshift relationship within the  $\Lambda$ CDM model for a flat universe. With  $\rho_M$  and  $\rho_\Lambda$  being the density of matter (ordinary and non-luminous) and of dark energy, the traditional Friedmann equation is being recast as [51]

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} (\rho_M + \rho_\Lambda) = H_0^2 \left[ \Omega_M \frac{1}{a^3} + \Omega_\Lambda \right] \quad (124)$$

in which

$$\Omega_M \triangleq \frac{\rho_M(t_0)}{\rho_c(t_0)} = \frac{8\pi G}{3H_0^2} \rho_M(t_0) \quad (125)$$

$$\Omega_\Lambda \triangleq \frac{\rho_\Lambda(t_0)}{\rho_c(t_0)} = \frac{8\pi G}{3H_0^2} \rho_\Lambda \quad (126)$$

$$\Omega_M + \Omega_\Lambda = 1 \quad (127)$$

Note that the density of ordinary matter and non-luminous matter scales inversely with volume,  $\rho_M \propto a^{-3}$ , whereas the density of dark energy is a constant.

For the almost flat space, setting  $k = 0$  in Eq. (104) and denote  $r$  for  $r_e$ , we obtain the proper distance from the galaxy and the Earth

$$r \approx c_0 \int_{t_e}^{t_o} \frac{dt}{a(t)}. \quad (128)$$

Utilizing the traditional Lemaitre redshift formula (111):

$$1 + z = a^{-1}(t) \Rightarrow dz = -\frac{\dot{a}}{a^2} dt \Rightarrow \frac{dt}{a} = -\frac{dz}{\dot{a}/a} \quad (129)$$

we obtain

$$r = c_0 \int_0^z \frac{dz'}{(\dot{a}/a)(z')}. \quad (130)$$

By virtue of Eqs. (124) and (111), we then get the traditional distance-redshift relationship

$$\frac{r}{c_0} = \frac{1}{H_0} \int_0^z \frac{dz'}{\sqrt{\Omega_M(1+z')^3 + \Omega_\Lambda}} \quad (131)$$

For low  $z$ , it recovers the Hubble law (123) as expected:

$$\frac{r}{c_0} \approx \frac{z}{H_0}.$$

Two special cases of interest:

- The Einstein-de Sitter universe ( $\Omega_M = 1, \Omega_\Lambda = 0$ ) corresponds to

$$\frac{r}{c} = \frac{2}{H_0} \left( 1 - \frac{1}{\sqrt{1+z}} \right) \quad (132)$$

- The de Sitter universe ( $\Omega_M = 0, \Omega_\Lambda = 1$ ) corresponds to

$$\frac{r}{c} = \frac{z}{H_0} \quad (133)$$

### 8.4.2 For curvature-scaling gravity:

Let us derive the distance-redshift relation in our approach. To proceed, we shall only need the following result, which is to be derived in Section 9. The evolution of the cosmic scale factor, derived solely from the scaling rule of time duration, is found to be (see Eq. (169))

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3} \quad (134)$$

with  $a(t_0)$  being set equal 1. Taking derivative of the above equation, we obtain the Hubble constant

$$H(t) \triangleq \frac{\dot{a}(t)}{a(t)} = \frac{2}{3t} \quad (135)$$

from which, the age formula is thus

$$t_0 = \frac{2}{3H_0}. \quad (136)$$

We further have

$$\frac{\dot{a}}{a} = H = \frac{2}{3t} = \left(\frac{2}{3t_0}\right) \left(\frac{t_0}{t}\right) = H_0 \frac{1}{a^{3/2}}. \quad (137)$$

With Eq. (114) for  $k = 0$ , the proper distance in the almost flat space is

$$r \approx \int_{t_e}^{t_o} \frac{c_0 dt}{a^{3/2}(t)}. \quad (138)$$

Utilizing the modified Lemaitre redshift formula (118):

$$1 + z = a^{-3/2}(t) \Rightarrow dz = -\frac{3}{2} \frac{\dot{a}}{a^{5/2}} dt \Rightarrow \frac{dt}{a^{3/2}} = -\frac{2}{3} \frac{dz}{a^{5/2}} \quad (139)$$

we obtain

$$r = \frac{2}{3} c_0 \int_0^z \frac{dz'}{(\dot{a}/a)(z')} \quad (140)$$

By virtue of Eqs. (137) and (118), we then get the modified distance-redshift relationship

$$\frac{r}{c_0} = \frac{2}{3H_0} \int_0^z \frac{dz'}{1+z'} = \frac{2}{3H_0} \ln(1+z) \quad (141)$$

Two important features to note:

- Only  $H_0$  participates in (141). The mass density and the spatial curvature are not involved. This helps make the fitting to observational data parsimonious and universal.
- The correction factor of  $\frac{3}{2}$  for  $H_0$  is again manifest in the right-hand-side of (141). For low  $z$ , it recovers the modified Hubble law (122) as expected,  $\frac{r}{c_0} \approx \frac{2}{3H_0} z$ .

## 8.5 The modified luminosity-redshift relationship

So far we focused on the proper distance for cosmological objects. However, proper distance is not directly measurable but must be deduced from angular distance or luminosity distance. We shall work out the conversion from the proper distance to the luminosity distance in our approach.

### 8.5.1 For the Lambda-CDM model:

Consider a source located at the comoving coordinate  $\chi_e$  with total luminosity  $L$ . The energy output at emission time  $t_e$  within the window  $\delta t_e$  is given by

$$\Delta E_e = L \delta t_e. \quad (142)$$

As the photons traversed the distance, their energy gets “Doppler thieved” by the scale factor  $a(t_o)/a(t_e)$ . Therefore, at the moment of observation, the observed energy will be

$$\Delta E_o = \Delta E_e a(t_e). \quad (143)$$

The physical area of the sphere centered at  $r_e$  and radius  $r(z)$  to be crossed by photons today is

$$S(z) = 4\pi r^2(z). \quad (144)$$

The visible brightness (energy flux at observer's position) equals

$$J = \frac{\Delta E_o}{S(z) \Delta t_o} = \frac{\Delta E_e a(t_e)}{S(z) \Delta t_o} = \frac{(L \Delta t_e) a(t_e)}{S(z) \Delta t_o} = \frac{L}{S(z)} a^2(t_e) \quad (145)$$

in which we have made use of (106)

$$\frac{\delta t_o}{a(t_o)} = \frac{\delta t_e}{a(t_e)}.$$

Utilizing the traditional Lemaitre redshift formula (111), we obtain

$$J = \frac{L}{4\pi r^2(z)} (1+z)^{-2}. \quad (146)$$

The photometric distance  $d_L$  is defined via

$$J = \frac{L}{4\pi d_L^2(z)} \quad (147)$$

which leads to

$$d_L = (1+z) r(z). \quad (148)$$

Coupling with (131)

$$\frac{r}{c_0} = \frac{1}{H_0} \int_0^z \frac{dz'}{\sqrt{\Omega_M(1+z')^3 + \Omega_\Lambda}}, \quad (149)$$

we obtain the traditional luminosity-redshift relationship

$$\frac{d_L}{c_0} = \frac{1+z}{H_0} \int_0^z \frac{dz'}{\sqrt{\Omega_M(1+z')^3 + \Omega_\Lambda}} \quad (150)$$

For the sake of later comparison, the Einstein-de Sitter universe ( $\Omega_M = 1$ ,  $\Omega_\Lambda = 0$ ) has

$$\frac{d_L}{c_0} = 2 \frac{1+z}{H_0} \left( 1 - \frac{1}{\sqrt{1+z}} \right) \quad (151)$$

### 8.5.2 For curvature-scaling gravity:

In our approach, although the speed at which photons arrive at the observer gets modified, the Planck constant also gets modified. With  $c \propto a^{-1/2}$  and  $\hbar \propto a^{1/2}$ , the photon energy continues to scale as  $E = \hbar c / \lambda \propto a^{-1}$ . Therefore, the photon energy still gets ‘‘Doppler thieved’’ by the scale factor  $a(t_o)/a(t_e)$  as in the traditional approach.

The visible brightness (energy flux at observer's position) equals

$$J = \frac{\Delta E_o}{S(z) \Delta t_o} = \frac{\Delta E_e a(t_e)}{S(z) \Delta t_o} = \frac{(L \Delta t_e) a(t_e)}{S(z) \Delta t_o} = \frac{L}{S(z)} a^{5/2}(t_e) \quad (152)$$

in which we have made use of (116)

$$\frac{\delta t_o}{a^{3/2}(t_o)} = \frac{\delta t_e}{a^{3/2}(t_e)}.$$

Utilizing the modified Lemaitre redshift formula (118), we obtain

$$J = \frac{L}{4\pi r^2(z)} (1+z)^{-5/3}. \quad (153)$$

The photometric distance, defined in (147), now becomes

$$d_L = (1+z)^{5/6} r(z). \quad (154)$$

Coupling with (141)

$$\frac{r}{c_0} = \frac{2}{3H_0} \ln(1+z) \quad (155)$$

we obtain the modified photometric distance-redshift relationship

$$\frac{d_L}{c_0} = \frac{2}{3} \frac{(1+z)^{5/6}}{H_0} \ln(1+z) \quad (156)$$

This is the central formula of our curvature-scaling gravity approach to re-assess the Type Ia supernovae data. It is universally applicable to all shapes of the universe (flat/open/closed).

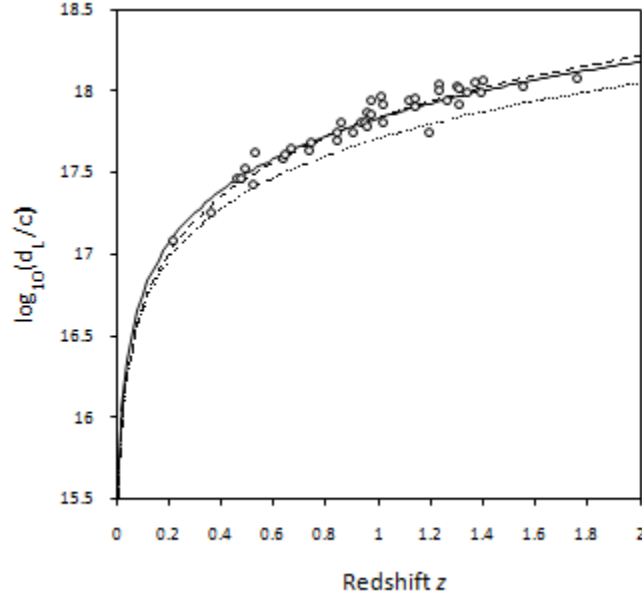


Fig. 1: Comparison of various luminosity distance-redshift formulae to Type Ia supernovae data. Open circles: 41 data points of high- $z$  objects listed in Table 4 of Ref [60]. Long-dashed line:  $\Lambda$ CDM model's formula, Eq. (150), with  $H_0 = 70.5$ ,  $\Omega_M = 0.27$ ,  $\Omega_\Lambda = 0.73$ . Dotted line: Einstein-de Sitter model's formula, Eq. (151), with  $H_0 = 70.5$  ( $\Omega_M = 1$ ,  $\Omega_\Lambda = 0$ ). Solid line: our formula, Eq. (156), using  $H_0 = 37.4$ .

## 8.6 A critical analysis of Type Ia supernovae data

In 1998, it was announced that Type Ia supernova of high-redshift reveal a surprising behavior [58, 59]. The supernovae appear fainter – and thus farther – than would have been expected for cosmological objects of these redshifts based on the Friedmann model. Type Ia supernovae are objects with stable luminosity, hence the name standard candles. Astronomers exploit this property to deduce the luminosity distance  $d_L$  from Earth of these objects. Their redshift values  $z$ , as usual, are independently obtained from the Doppler effect. For Type Ia supernovae, the curve  $d_L$  against  $z$  bends upward toward the high- $z$  end. This striking behavior of Type Ia supernovae is said to be the evidence for dark energy, which is modeled as the cosmological constant term  $\Omega_\Lambda$  in the  $\Lambda$ CDM model.

Standard cosmology, however, does not take into account the variation of light speed as the universe expands. We shall show in this subsection and the next that it is this very neglect in standard cosmology that is responsible for behavior of the Type Ia supernovae. We must reexamine the observational data using the revised redshift formulae derived in this section so far. The centerpiece in our analysis below is the modified photometric distance-redshift relationship (156):

$$\frac{d_L}{c_0} = \frac{2}{3} \frac{(1+z)^{5/6}}{H_0} \ln(1+z).$$

Observational data are cited in the form of distance modulus  $m - M$  which is related to photometric distance  $d_L$  as:

$$m - M = 5 \log_{10}(d_L/\text{Mpc}) + 25 \quad (157)$$

We retrieve the observational data of 41 high- $z$  objects from Table 6 of [60]. The values of  $z$  and of  $m - M$  are listed in columns 2 and 3 in the Table respectively. We then extract the photometric distances  $d_L$  via (157).

Figure 1 is the main result of this section. The vertical axis measures  $\log_{10}(d_L/c)$  where  $c$  is the value of speed of light measured on Earth (i.e., 300,000 km.sec<sup>-1</sup>). The set of 41 Type Ia supernovae data are shown in open circles. The solid line shows our model's Formula (156) corresponding to the only parameter  $H_0 = 37.4$  that best fits to the data. The long-dashed line corresponds to the  $\Lambda$ CDM model's Formula (150) with 3 parameters  $H_0 = 70.5$ ,  $\Omega_M = 0.27$ ,  $\Omega_\Lambda = 0.73$  (besides  $\Omega_{curv} = 1 - \Omega_M - \Omega_\Lambda = 0$ ). The dotted line shows Einstein-de Sitter universe's Formula (151) using  $H_0 = 70.5$ .

The mean average error (MAE) between a model's prediction and the 41 observational data is:

$$\frac{1}{41} \sum_{i=1}^{41} \left| \log_{10} \frac{d_L^{\text{model}}(i)}{c} - \log_{10} \frac{d_L^{\text{observed}}(i)}{c} \right| \quad (158)$$

The MAE of our Formula (156) is 0.0469 which is comparable to the MAE of the  $\Lambda$ CDM model (150), 0.0476.

The most striking result is a new – and much lower – value for Hubble constant  $H_0 = 37.4$ . The reduction in value of  $H_0$  (from the conventionally accepted 70.5) leads to two important consequences:

1. A reduction in value of the critical density:  $\rho_c = 3H_0^2/(8\pi G)$  is reduced by the factor  $(70.5/37.4)^2 \approx 3.54$ . That means the actual  $\rho_c$  should be only 28% of the long-believed value. Interestingly, the new value of  $\rho_c$  approximately equals the total amount of ordinary matter and “dark matter” found in the universe.<sup>28</sup> The budgetary shortfall in density (to maintain a near-flat space) disappears. “Dark energy” is not needed.
2. The universe age: Our universal age formula (136),  $t_0 = \frac{2}{3H_0}$ , yields 17.4 Gly. The overestimated value of 70.5 for  $H_0$ , on the other hand, led to a much younger age at 9.3 Gly if one assumes the Einstein-de Sitter model. Such a young universe would be at odd with the established age of the oldest stars.<sup>29</sup>

These two results will throw important light onto the orthodox interpretation of an “accelerating” universe discussed in the following section.

## 8.7 An alternative interpretation to Type Ia supernovae data based on curvature-scaling gravity

A widely accepted view in standard cosmology is that the universe expansion has been accelerating. The conclusion was inferred from the data of Type Ia supernovae [58, 59]. In these ground-breaking discoveries, distant supernovae appear substantially fainter than what would have been expected for objects at the same redshift (if the cosmological term is absent). This in turn indicates that the supernovae were farther than what the standard model would have predicted for their redshift. An explanation is that the space expansion at the time was slower than expected from the standard Friedmann model, or, in other words, the expansion has been speeding up; hence the name “accelerating” expansion. This interpretation has fueled both observational and theoretical searches for a form of “dark energy” that supposedly speeds up the cosmic expansion.

As we pointed out in this section so far however, the Friedmann model fails to take into account the dependence of  $c$  on the cosmic scale  $a$ . As such, the redshift formula (150) is incorrect and thus cannot be used to analyze the Type Ia supernovae data. The conclusion regarding “acceleration expansion” is based on this flawed formula. Instead, the correct redshift formula is (156) which involves only one parameter  $H_0$ . The reported value of  $H_0 \approx 70.5$  was overestimated inadvertently from the flawed redshift formula. Using the correct formula (156), the reduced value for  $H_0 \approx 37.4$  simultaneously accounts for three things: (i) The universe’s correct age of 17.4 Gly, (ii) The budgetary shortfall (the correct critical density being only 28% of what previously thought), and (iii) The fit to the Type Ia supernovae data – a fit as good as the  $\Lambda$ CDM model produced (see Fig. 1). Again, our resolution involves only  $H_0$  and does not resort to any fudge factors whatsoever, such as the amount of “dark energy”.

Qualitatively, there are two intuitive interpretations for the observational data of Type Ia supernovae:

1. Consider two supernovae  $A$  and  $B$  at distances  $3bn$  and  $6bn$  light years away from the Earth, viz.  $d_B = 2d_A$ . Standard cosmology dictates the redshift values of  $z_A$  and  $z_B \approx 2z_A$  for them (to first-order approximation.) However, light travelled faster in a distant past than it did in a more recent epoch. Thus, the  $B$ -photon covered twice as long the distance in less than twice the amount of time as compared with the  $A$ -photon. Having spent less time in transit than expected, the  $B$ -photon experienced less cosmic expansion than expected, and thus less redshift than standard cosmology predicts. Namely:  $z_B < 2z_A$ . This result means an upward slopping in the curve as we put the two supernovae’s data on the  $d$  vs.  $z$  plot. Conversely, a supernova  $C$  with  $z_C = 2z_A$  must correspond to a distance greater than  $6bn$  light years, viz.  $d_C > d_B = 2d_A$ , and thus is a fainter object. This is precisely what is observed in Type Ia supernovae [58, 59].
2. Consider the Hubble law which is a good first-order approximation for the distance-redshift relationship. (Although supernovae billions light years away from the Earth do not exactly obey the Hubble law, the Hubble law is used in this paragraph as a baseline to illustrate the logic behind the deviation from the traditional distance-redshift

<sup>28</sup> We must note that although the total 28% leftover is said to be consisted of ordinary matter and “dark matter”, in light of our consideration in Section 4, the “dark matter” component is not necessarily a hypothetical invisible form of matter. It could be nothing but the linear Mannheim-Kazanas potential term (for ordinary matter) which, after adding up ordinary matter in the universe altogether, acts as surrogate for the “dark matter” content.

<sup>29</sup> A standard resolution for the age problem is said to have been found in [61] via the  $\Lambda$ CDM model’s age formula:  $t_0 = (2 \tanh^{-1} \sqrt{\Omega_\Lambda}) / (3H_0 \sqrt{\Omega_\Lambda})$  which yields 13.8 Gly using  $H_0 = 70.5$ ,  $\Omega_M = 0.27$ ,  $\Omega_\Lambda = 0.73$ . For completeness, the  $\Lambda$ CDM model’s age formula is derived below. In a flat space, using Eq. (124),  $\frac{a}{a} = H_0 (\Omega_M a^{-3} + \Omega_\Lambda)^{\frac{1}{2}}$ , we get

$$t_0 = \frac{1}{H_0} \int_0^1 \frac{da}{a (\Omega_M a^{-3} + \Omega_\Lambda)^{1/2}} = \frac{2}{3H_0} \frac{\tanh^{-1} \sqrt{\Omega_\Lambda}}{\sqrt{\Omega_\Lambda}}. \quad (159)$$

The  $\Lambda$ CDM model assumes an acceleration phase accompanying a deceleration phase. With  $\Omega_\Lambda = 0.73$ , Eq. (159) yields  $t_0 \approx H_0^{-1}$ . In light of the overestimation of the Hubble constant in standard cosmology, the age problem is resolved in our approach. The  $\Lambda$ CDM answer [61] based on dark energy is no longer necessary.

formula.) The farther the object, the higher its redshift. Per Hubble law, its redshift  $z$  and distance to Earth  $d$  are in linear proportion <sup>30</sup>:

$$z = \frac{H}{c} d + \text{higher-order terms} \quad (160)$$

or

$$d \approx z \frac{c}{H} \quad (161)$$

where  $H$  is the rate of expansion at redshift  $z$ . The value of  $z$  is deduced from the photon's redshift. The value of  $d$  is deduced from the luminosity of the object. Prior to 1998, the supernova at a given  $z$  had been expected to be at a distance  $d$  dictated by Eq. (161). The fainter-than-expected supernovae observed in 1998 [58] were thus at a greater distance than given in (161). The conventional explanation then holds that for a given  $z$ , the higher  $d$  than usual implies a value of  $H$  lower than usual. So, the supernovae must have experienced a lower rate of expansion  $H$  than now. One then concluded that the universe has been expanding faster and faster.

Curvature-scaling gravity offers an alternative interpretation: for a given  $z$ , the higher  $d$  than usual implies a value of  $c$  higher than usual. So, light in the past must have traveled faster than it does now. This conclusion dovetails perfectly with curvature-scaling gravity that light speed has been adapting to the lowering Ricci scalar, per (58):  $c \propto |\mathcal{R}|^{1/4}$ , as the universe expands. The universe is not accelerating.

Note that we deliberately avoided the “light-slows-down” misnomer since the jargon fails to reflect the right physics: in curvature-scaling gravity,  $c$  adapts to the Ricci scalar (or, equivalently, to the cosmic scale factor); the variation in  $c$  is not as a result of the cosmic time passage.

If one had not prejudiciously subscribed to the seven-decades-old Friedmann model and an (overreaching) belief on a universal  $c$ , the ground-breaking discoveries regarding Type Ia supernovae in 1998 [58, 59] would have squarely been seen as compelling evidence in favor of the variation of  $c$  as the universe expands. As such, these discoveries carry far more profound impacts than so far thought because they help get rid of the once-sacred universality of light speed from the scientific vocabulary. Such a conclusion is far more significant and valuable than what ad hoc addenda such as the “accelerating expansion” and “dark energy” could bring forth. These addenda were unnecessary supplementaries trying to bring the observational data back in line with the problematic Friedmann model which neglects the variation of  $c$  in its treatment.

---

<sup>30</sup> As we clarified by now, there is an additional factor of  $\frac{3}{2}$  in the Hubble law above, but this factor is not relevant for our reasoning here.

## 9 Implications of curvature-scaling gravity in cosmology

As we elaborated in the preceding section, the standard Friedmann model does not take into account the variation of light speed as function of the Ricci scalar (viz. the cosmic scale factor.) This neglect is the root of the age problem and the deviation from the standard redshift formulae, the deviation observed in Type Ia supernovae [58, 59]. Upon incorporating the  $c$ -variation, we are able to resolve these problems from first principles. Beside those two problems, the Friedmann model encountered other serious difficulties, both observationally and theoretically. Observationally, the Friedmann model standing alone could not explain the near uniform horizon and the near flatness of space. Theoretically, it suffered a series of cosmic coincidences (also known as fine tunings), such as Dicke's instability (viz. the oldness problem) and the budgetary shortfall. To reconcile the Friedmann model with observations as well as to salvage the theoretical basis of the Friedman model itself, standard cosmology has augmented the model with supplementaries which require new physics. The supplementaries are the dark energy and the inflationary expansion hypothesis. The supplementaries, however, are plagued with their own fine tuning problems and difficulties.

In this section, we shall apply curvature-scaling gravity, which allows the variation of  $c$  as function of the cosmic scale factor, to resolve some of the most pressing problems in cosmology: (i) the horizon problem, (ii) the flatness problem, and (ii) the oldness problem. Our results are derived solely from the scaling rule (58),  $c \propto a^{-1/2}$ , and are model-independent; that is to say, they do not rely on any model for the distribution of matter in the universe, apart from the homogeneity and isotropy assumptions of space.

### 9.1 The evolution of the cosmic scale

Our treatment of cosmology is based on the modified Robertson-Walker metric which expressly allows the dependence of  $c$  on the cosmic scale factor. The modified RW metric was given in Eq. (101):

$$\begin{aligned} ds^2 &= c_0^2 \frac{a_0}{a(t)} dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right] \\ &= a^2(t) \left[ c_0^2 \frac{a_0}{a^3(t)} dt^2 - \frac{dr^2}{1 - kr^2} - r^2 d\Omega^2 \right] \end{aligned} \quad (162)$$

in which we restore  $a(t_0) = a_0$ . Closed, flat, open universe corresponds to  $k > 0$ ,  $k = 0$ ,  $k < 0$  respectively. Further define the conformal time in terms of the cosmic time  $t$  via

$$d\eta = c_0 \frac{a_0^{1/2}}{a^{3/2}(t)} dt. \quad (163)$$

The modified RW metric becomes

$$ds^2 = a^2(\eta) \left[ d^2\eta - \frac{dr^2}{1 - kr^2} - r^2 d\Omega^2 \right]. \quad (164)$$

The rate of change for the cosmic scale  $a(t)$  is determined solely by the current state of the universe. At the scale factor  $a$ , the rate of change should be a function of  $a$  only, namely

$$\frac{da}{dt} = f(a) \quad (165)$$

with  $f$  being a functional form to be specified. With the scaling rule for time duration (56), we have

$$da \propto a \quad (166)$$

$$dt \propto a^{3/2}. \quad (167)$$

As such

$$\frac{da}{dt} = \frac{\text{const}}{a^{1/2}}. \quad (168)$$

This equation accepts a solution

$$a(t) = a_0 \left( \frac{t}{t_0} \right)^{2/3} \quad (169)$$

with  $a_0$  being the current scale factor. The Hubble constant then is

$$H \triangleq \frac{\dot{a}}{a} = \frac{2}{3t}. \quad (170)$$



The current age of the universe is related to the current value of the Hubble constant:

$$t_0 = \frac{2}{3H_0}, \quad (171)$$

from which we obtain the evolution rule for the cosmic scale factor

$$a(t) = a_0 \left( \frac{3}{2} H_0 t \right)^{2/3}. \quad (172)$$

These results are entirely generic, independent of any specific details of the matter distribution or content in universe, apart from the homogeneity and isotropy of space.

The universe is thus expanding in the critical fashion:  $a \simeq t^{2/3}$ , regardless of the amount of matter and/or the nature of matter content (radiation, fermionic, etc.) and the shape of the Universe (open/closed/flat). Unlike the Friedmann equations which typically allow 3 different modes <sup>31</sup>, in our approach, the evolution is unique and robust; the “critical fashion” of the cosmic expansion is also precisely what is seen in observational cosmology.

Another piece of evidence in support of the evolution (172) can also be seen as follows. From (163)

$$\frac{d\eta}{dt} = c_0 \frac{a_0^{1/2}}{a^{3/2}} = \frac{c_0}{a_0} \left( \frac{a_0}{a} \right)^{3/2} = \frac{c_0}{a_0} \frac{t_0}{t} \quad (173)$$

and (171), we get

$$t = t_0 \exp \left( \frac{a_0}{c_0 t_0} \eta \right) = t_0 \exp \left( \frac{3}{2} \frac{H_0 a_0}{c_0} \eta \right) \quad (174)$$

with  $\eta = 0$  corresponding to today  $t_0$ . Combining with (169), we get

$$a(\eta) = a_0 \exp \left( \frac{H_0 a_0}{c_0} \eta \right) \quad (175)$$

Next, for the modified RW metric, the Ricci scalar is (see Appendix E)

$$\mathcal{R} = -\frac{6}{a^2} \left( \frac{1}{a} \frac{d^2 a}{d\eta^2} + \kappa \right). \quad (176)$$

and, by virtue of (175)

$$\mathcal{R} = -6 \frac{H_0^2 a_0^2 c_0^{-2} + \kappa}{a^2}. \quad (177)$$

The Ricci length then reads

$$a_{\mathcal{R}} \triangleq |\mathcal{R}|^{-1/2} = \frac{a}{\sqrt{6 |H_0^2 a_0^2 c_0^{-2} + \kappa|}}$$

which scales precisely as the cosmic scale factor. An evolution rule away from (169) would likely jeopardize the proportion between the Ricci length and the cosmic scale factor.

## 9.2 Resolution to the age problem

Our resolution to the universe’s age has been discussed in details in Sections 8.6 and 8.7. Here, we only wish to recap. The age formula (171)

$$t_0 = \frac{2}{3H_0}$$

is robust. It is derivable solely from the scaling property of time duration. Furthermore, it is valid for all shapes of the universe.

With the corrected  $H_0 = 37.4$  obtained in Section 8.6, the universe age is  $t_0 = 17.4 Gy$  comfortably accommodating its oldest stars. The standard  $\Lambda$ CDM resolution [61] that invokes an acceleration phase following by a deceleration phase is no longer necessary.

### The past and future of the universe:

Regardless of its shape and/or density, the universe always grows in accordance with a universal law (169):  $a \propto t^{2/3}$ . The universe will not collapse nor expand supercritically. Note that the mode of expansion in the Friedmann model is highly sensitive to the initial state of the universe. To explain the observed critical mode, the Friedmann model would require a fine tuning to an extraordinary level. The anthropic principle is not needed in our approach.

<sup>31</sup> Plus the acceleration mode induced by the cosmological constant.

### 9.3 Resolution to the horizon problem

The cosmic microwave background (CMB) observational result shows a highly uniform distribution (to the accuracy of  $10^{-5}$ ) of cosmic radiation across the horizon. This uniformity presents a serious challenge to the Friedmann model. The model needs to reconcile the observed uniformity with the fact that the current horizon is not causally connected. However, as we pointed out before, the Friedmann model neglects the variability of  $c$  as function of the cosmic scale factor  $a$ . As the universe expands,  $c$  gradually drops accordingly in reverse proportion to  $\sqrt{a}$ . Conversely, as we trace back to the origin of the cosmic time,  $c$  was higher when the universe was smaller.

More concretely, in the modified RW metric, the cosmological horizon is given by:

$$l_H(t_0) = \int d\eta = \int_0^{t_0} \frac{c_0 a_0^{1/2} d\tau}{a^{3/2}(\tau)} \quad (178)$$

in which we utilized the conversion (163). Since  $a(\tau) = a_0 \left(\frac{\tau}{t_0}\right)^{2/3}$  per (169), we obtain

$$l_H(t_0) = \int_0^{t_0} \frac{c_0 a_0^{1/2} d\tau}{a_0^{3/2} (\tau/t_0)} = \frac{c_0 t_0}{a_0} \int_0^{t_0} \frac{d\tau}{\tau} = \infty. \quad (179)$$

The cosmological horizon is (logarithmically) divergent. Thus, the entire universe was causally connected. This result neatly helps explain the near uniformity in our current horizon. Our explanation comes purely from the scaling rules and the modified RW metric. Interestingly, the divergence in (179) is logarithmic and is thus parsimonious – a rather curious result.

Our treatment allows  $c$  to vary with the scale factor and, otherwise, is in line with Moffat, and Magueijo and Albrecht's original insights in resolving the horizon problem using variable speed of light mechanism. In [52], Moffat stipulated that if the velocity of light in the baby universe was higher than it is now by a factor of  $10^{30}$ , the horizon of the early universe would be in causal contact and thus achieve uniformity. Beside Moffat's original proposal that  $c$  underwent a first-order phase transition from a very high value to its current value, subsequent developments of Moffat, and of Magueijo and Albrecht [53, 54] allowed a smooth variation in  $c$ . Unlike these author's approaches, our approach does not require the need to devise a dynamics for the variable  $c$ . In our work,  $c$  is dependent on the cosmic scale factor instead.

To explain the horizon problem, standard cosmology augmented the Friedmann model with an inflationary phase in which the baby universe underwent an exponential expansion mode [56]. Before the inflationary phase, the whole horizon was speculated to have been in causal contact and have thus reached a thermal equilibrium. As the inflation kicked in, different sections in the horizon were rapidly pulled away from one another and, as a result, now appear separated. The inflationary universe hypothesis in turn requires several ad hoc assumptions to explain the mechanism of the inflation and the nature of the agent that could cause it. These assumptions are not intrinsic to the established physics. Furthermore, the hypothesis encounters its own fine-tuning problems.

Curvature-scaling gravity provides a natural explanation to the horizon problem, without any ad hoc assumptions. The underlying agent is the adaptation of  $c$  to the prevailing Ricci scalar along the course of the universe expansion.

### 9.4 Resolution to the flatness problem

The Wilkinson Microwave Anisotropy Probe (WMAP) has confirmed that the universe is essentially flat. The universe thus required specific mechanisms to achieve and sustain flatness.

Our model also offers a natural explanation to the flatness problem. As the early Universe expanded, the speed at which light travels decreased, according the scaling rule (58),  $c \propto a^{-1/2}$ . As the universe grew in size several orders over, at the same time the speed of light dropped precipitously several orders in magnitude. The effective horizon shrank relatively to the universe's size. By the decoupling event when light began to travel freely, only light from a pocket around us can reach us today. This explains the flatness problem. Note that the flatness problem is not identical to the horizon problem; even before the decoupling event, matter still could communicate via other channels, such as neutrinos or gravitons. The WMAP picture is available for light at the decoupling which occurred at the decoupling event which is believed to have occurred much later after the Big Bang. Our approach is able to provide a unified and simultaneous explanation for both, the flatness problem and the horizon problem.

Our explanation should be put in comparison with the inflationary universe hypothesis which stipulated an exponential expansion in the baby universe to help flatten out the universe regardless of its initial state. In our approach, no new elements or assumptions are needed to account for the flatness. Also, the universe had ample time – from the Big Bang to the decoupling event – to achieve flatness.

## 9.5 Resolution to Dicke’s “runaway density parameter” problem or the oldness problem

Let us first review the situation within the traditional framework. From the traditional Friedmann equation [30, 31]:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k c^2}{a^2} \quad (180)$$

and the critical density defined as  $\rho_c = 3H^2/(8\pi G)$ , the density parameter

$$\Omega \triangleq \frac{\rho}{\rho_c} \quad (181)$$

satisfies

$$(\Omega^{-1} - 1) \rho a^2 = -\frac{3kc^2}{8\pi G} \quad (182)$$

The right-hand-side is a constant, whereas  $\rho a^2$  scales as  $a^{-1}$ . To compensate the drop in  $\rho a^2$  as  $a$  grows,  $|\Omega^{-1} - 1|$  must grow and  $\Omega$  is driven away from 1 if it started away from 1. This spells the instability trouble for  $\Omega$ . The runaway of  $\Omega$  from 1 has been known to be rapid. In order to account for a relatively flat space at our current era, the universe must have started out extremely flat and the value for  $\Omega$  at recombination time must have been fine-tuned to unity at high precision. The “runaway” problem posed a major challenge to the Friedmann model.

Our approach does not suffer this problem. That is because both  $\rho$  and  $\rho_c$  scale similarly. Whereas  $\rho \propto a^{-3}$ , the critical density scales as

$$\rho_c \propto H^2 = \left(\frac{2}{3t}\right)^2 \propto a^{-3}. \quad (183)$$

As a result,  $\Omega = \text{const}$  at all time points.

In hindsight, the reason behind the “runaway” problem in the traditional framework was the scaling mismatch in the Friedmann equation:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k c^2}{a^2} \quad (184)$$

in which the three terms scale differently. Whereas  $\rho \propto a^{-3}$ , the last term scales as  $a^{-2}$  and how the first term scales depends on the shape of space. This scaling mismatch then got transferred to Eq. (182), creating the dreaded instability problem. One tentative way to fix this problem for the Friedmann equation is to allow  $c$  in the last term in (184) to scale as  $a^{-1/2}$ , per the scaling rule (58). Note that  $c$  depends on  $a$ , instead of  $t$ . As such, the replacement  $c = c_0 a^{-1/2}$  is legitimate. In so doing, the right-hand-side of (184) scales as  $a^{-3}$ , and the equation subsequently yields  $a \propto t^{2/3}$  in perfect agreement with the evolution rule (169) in our curvature-scaling gravity approach.

## 9.6 On the shortcomings of the Friedmann model and its related supplementaries

We offered a new look at cosmology in the light of curvature-scaling gravity. The Universe evolves in a unique and robust fashion, independent of its shape and its matter content and density. As the universe expands, its curvature gradually drops, leading to a drop in  $c$  in a precise connection to  $\mathcal{R}$  per (58),  $c \propto |\mathcal{R}|^{1/4}$ . With only this property and the RW metric, we are able to explain the observational data of Type Ia supernova, the age problem, the flatness problem, the horizon problem. The future task remains to come up with a reasonable model for the stress-energy tensor for matter in the universe. Any new model in this line of research based on curvature-scaling gravity will call for a recalibration of observational data, such as the WMAP.

Let us further comment on an important consequence of curvature-scaling gravity: the variability of  $\hbar$  as function of the universe’s Ricci scalar. In the early stage of the universe,  $\mathcal{R}$  was higher leading to smaller values for  $\hbar$  according to the scaling rule (57):  $\hbar \propto |\mathcal{R}|^{-1/4}$ . Quantum effects were decidedly weaker in the early universe. As a result, any new model of cosmology will need a recalibration to observational data, such as the WMAP data for the acoustic peaks, for example. In another example, the decoupling event is said to have occurred at about 380,000 years after the Big Bang. Due to the variation of  $\hbar$  (which governs the coupling of matter with photons), the precise value of this time point need recalibration<sup>32</sup>.

The standard Friedmann model does not allow the variability of  $c$  as a function of the cosmic scale factor  $a$  (or, equivalently, the Ricci scalar which has been falling as the universe expands.) It is thus a problematic cosmological model to start with. The model in its original form could not account for several observational facts and data. It produced flawed formulae for the redshift and the Hubble law, leading to an upward biased estimate of  $H_0$ . These problems appear

<sup>32</sup> As an aside note, absence of monopoles was first credited as one of three motivations, the flatness problem and the horizon problem rounding out the list, for the inflation hypothesis. It was said that an inflation diluted the monopoles. If the field-theoretical models for monopoles are indeed correct, the absence of monopoles can be naturally explained in curvature-scaling gravity. The particle physics models that predicted an abundant amount of monopoles used the current value of  $\hbar$ . These models should have used smaller value of  $\hbar$ .

in cosmology since the cosmos is a laboratory which accumulates the effects of non-universal  $c$  over a lifespan of billions light years. Most severe cracks show up in the age problem (via the overestimated Hubble constant), the deviation in Type Ia supernovae, the horizon problem, the flatness problem, and other cosmic coincidences. To resolve these problems, standard cosmology had to resort to a series of ad hoc solutions, compiled in the concordance  $\Lambda$ CDM model. In the light of curvature-scaling gravity, these ad hoc solutions are unnecessary.

1. The lack of necessity of accelerating expansion:

As we elaborated in Sections 8.6 and 8.7, the data of Type Ia supernovae can be naturally explained in curvature-scaling gravity which allows the light speed to vary with respect to the cosmic scale factor per  $c \propto a^{-1/2}$ . The cosmic expansion is not accelerating. In light of the misperceived meaning of the Michelson-Morley experiment, the discoveries regarding Type Ia supernovae [58, 59] should be seen as compelling evidence in support of the variation of  $c$ , a conclusion which would profoundly deepen one's understanding of spacetime and gravity. This is where the discoveries show their greatest importance and impacts.

2. The resolution to the budgetary shortfall and the lack of necessity for “dark energy”:

The overestimation of  $H_0$  will help address the budgetary shortfall problem as well. Whilst a more complete model of cosmology is required, let us take the original Friedmann equation (180)

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k c^2}{a^2}$$

and tentatively correct it by letting  $c$  depend on  $a$  per  $c = c_0 a^{-1/2}$  with  $c_0$  being the speed of light measure today in the outer space (which, unlike the galaxies, is subject to cosmic expansion). Note that  $c$  does not explicitly depend on  $t$ ; rather,  $c$  adapts to the scale factor  $a$  (or, equivalently, to the Ricci scalar). As such, the replacement  $c \rightarrow c_0 a^{-1/2}$  is legitimate, yielding the following equation:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k c_0^2}{a^3} \quad (185)$$

This equation adopts a unique evolution rule:  $a \propto t^{2/3}$  in full agreement with (169) in our curvature-scaling gravity approach<sup>33</sup>. With the WMAP confirming a flat space,  $k = 0$ , Eq. (185) admits a solution

$$a = (6\pi G \rho_c)^{1/3} t^{2/3} \quad (186)$$

in agreement with (169) in our curvature-scaling gravity approach. Combined (171) and (186), the current critical density satisfies

$$6\pi G \rho_c = \left(\frac{3}{2}H_0\right)^2, \quad (187)$$

thus remaining to be

$$\rho_c = \frac{3}{8\pi G} H_0^2. \quad (188)$$

Recall that  $H_0$  has been overestimated, however. With the value of  $H_0$  being corrected to 37.4 instead of 70.5, the critical density is reduced to  $\left(\frac{37.4}{70.5}\right)^2 \approx 28\%$  of the accepted value in standard cosmology. Interestingly, this reduced value of  $\rho_c$  almost precisely matches the amount of baryonic matter and dark matter in the Universe. The missing gap for the budget thus disappears; the hypothetical “dark energy” invented to make up the shortfall is not needed. The budgetary shortfall has often been cited as solid evidence in support of “dark energy”. This is no longer the case. The overestimated  $H_0$  was precisely the culprit for the shortfall in the matter budget<sup>34</sup>.

The cosmological constant problem and the coincidence of dark energy density of 0.72 at our current epoch are handily avoided.

<sup>33</sup> The fact that the original Friedmann equation does admit the critical expansion mode,  $a \propto t^{2/3}$  and yields the correct age formula,  $t = 2/(3H_0)$  for  $k = 0$  compatible with the generic scaling rule indicates that the Friedmann model does carry some element of truth and should continue to be valuable upon future revisions.

<sup>34</sup> Also note that although the 28% leftover is said to be composed of baryonic matter and dark matter, the dark matter component is not necessarily some hypothetical non-luminous form of matter, such as WIMPs. As pointed out in Section 4, the linear Mannheim-Kazanas term that is responsible for galactic rotation curves will continue to be a surrogate for the “dark matter” content for the Universe at large. Indeed, in Mannheim's theory [3–6], the parameters  $\gamma^*$  and  $\gamma_0$  are found to acquire universal values which act as a centripetal force toward the ordinary mass source. We could thus stipulate that luminous matter would be the only source of gravity, with the Mannheim-Kazanas potential playing a surrogate to the “dark matter” sector, and “dark energy” being eliminated.

### 3. The lack of necessity of inflationary expansion:

The Friedmann model cannot account for other observations – the uniformity of observable horizon and the flatness of space. The model furthermore suffers from its own theoretical problem – the oldness problem stemming from Dicke’s instability, the extraordinarily fine-tuned initial state so that it could evolve and survive until today. In light of curvature-scaling gravity, all these problems are non-existent. They only appeared to be problems if they were viewed from the basis of the Friedmann model.

To reconcile these observations with the Friedmann model, the inflationary universe hypothesis, which posited an exponential growth for the baby universe, was proposed in the early eighties [56] and later became integrated into standard cosmology. There has not been any direct evidence of inflation, to the best of our knowledge. It is, though, often said that the WMAP data are hallmarks of inflation. This is not necessarily so. Most of the WMAP results are generic features of a Big Bang theory.

The inflation hypothesis introduced a new set of fine tunings and problems on its own, the most notable of which are the slow-roll condition, the graceful exit, and the nature of the cosmological constant [57]. It was also particularly designed to address questions regarding the very early stage of the universe and has little to say about the subsequent development of the universe until our current epoch and about the future of the universe.

In summary, upon the reinterpretation in the light of curvature-scaling gravity that the value of  $c$  is not universal but rather a function of the Ricci scalar as the universe expands, the issues once considered fatal to the Friedmann model disappear. Any cosmological model to be devised in the future must allow  $c$ -variability. In this direction, curvature-scaling gravity already provides the guideline for any revision to come: the evolution of the cosmic factor obeys the scaling rule (58). The conclusions from curvature-scaling gravity presented in this section and the preceding one are model-independent; they must unequivocally hold for any theory of cosmology.

## 10 Conclusions and outlook of curvature-scaling gravity in quantum gravity

Einstein's general theory of relativity was the first methodical attempt to extrapolate the physics in man's living quarter to the entire universe. The theory seeks guidance in the equivalence principle and the general covariance principle that physical laws (including special relativity) are valid in the tangent frames local to each point on the manifold. Implicit in general relativity, though, is the assumption of a universal absolute length scale against which all processes and phenomena – gravitational and non-gravitational alike – are measured. Although general relativity offers one concrete construction of the spacetime manifold, we provide an alternative admissible construction of the manifold which satisfies all of Einstein's requirements with regard to relativity and causality.

To do so, we extend Einstein's guidance further: not only are physical laws locally valid in the tangent frames, the length scale for them is valid only locally. We inherit fully Einstein's vision of the Riemannian geometrical structure of the spacetime manifold and all other principles that he establishes (the Lorentz symmetry, the relativity principle, and the general covariance principle), whereas we further assign the Ricci scalar  $\mathcal{R}$  a new eminent role: in each local region on the manifold, the Ricci length, defined as  $a_{\mathcal{R}} \triangleq |\mathcal{R}|^{-1/2}$ , is the intrinsic length which all other lengths – including the Bohr radius of quantum mechanics, e.g. – are pegged onto. Intuitively speaking, the Ricci scalar determines the size of every object in a local region. Mathematically speaking, the action for any physical process is to be built from the dimensionless ratio of length and the invariant Ricci length  $a_{\mathcal{R}}$ .

The spacetime manifold is thus a patchwork of local regions, obeying two requirements (or postulates):

- (I) Special relativity and all physical laws (of non-gravitational origin) be satisfied in each local region;
- (II) Each region locally accept the intrinsic Ricci length as the fundamental length scale for all processes that take place within the region.

Requirement (I) is nothing but a reinforcement of the equivalence principle. Requirement (II) is the only new feature that sets our approach apart from Einstein's theory. From the two requirements, we were able to construct the spacetime manifold which respects causality globally, and obtain the Lagrangian of gravitational field coupled with matter.

In our approach, curvature is thereby promoted: not only does it determine the geometric structure of the underlying manifold, it is actively involved in the dynamics of physical processes by setting the scale for them. The intuitive rationale is that spacetime should provide the scale (in this case,  $a_{\mathcal{R}}$ ) for physical processes, instead of the other way around. The ramification of this initiative of ours is a series of conceptual departures:

1. The fundamental length scale  $a_{\mathcal{R}}$  for physical processes is itself a dynamical variable governed by the equations of the metric  $g_{\mu\nu}$ . As such, the gravitational field is deeply involved in the physical processes.
2. A new recipe to obtain the Lagrangian of gravity coupled with matter, away from the standard minimal coupling procedure. The action is to be constructed from dimensionless ratios of lengths using the Ricci length as the common denominator. Our recipe entails the following replacements:  $dx^\mu \rightarrow dx^\mu/a_{\mathcal{R}} = |\mathcal{R}|^{1/2} dx^\mu$ ;  $\nabla_\mu \rightarrow a_{\mathcal{R}} \nabla_\mu = |\mathcal{R}|^{-1/2} \nabla_\mu$ . As such, the volume element is replaced as  $d^4x \sqrt{-\det g} \rightarrow d^4x \mathcal{R}^2 \sqrt{-\det g}$ . Through our recipe, the gravitational field arises organically from the matter fields. (See Section 7.)
3. Anisotropy in the scaling of time: with Requirement (I) that physical laws retain their forms – but not necessarily their parameters – in every local region on the manifold, and Requirement (II) that the physical laws adapt to the prevailing Ricci length, the scaling of time duration is found (via the Schrödinger equation or the action of QED) to be anisotropic with respect to the Ricci length:  $dt \propto a_{\mathcal{R}}^{3/2}$  as compared to that of space,  $dx \propto a_{\mathcal{R}}$ . (See Section 5.1.)
4. We corrected a deep misperception that the Michelson-Morley finding necessarily meant a universal value of light speed everywhere. All Michelson-Morley's finding establishes is that the speed of light at each given location is the same regardless of the direction of the light beam and/or the motion of the observer. It has nothing to say about the equality (or the lack thereof) in the value of  $c$  at different locations. By virtue of the equivalence principle which is local in nature, Lorentz invariance, the Michelson-Morley result, and the relativity principle are also local in nature. The value of  $c$  is meant to be local; that is to say,  $c$  acquires a new value for each location on the manifold. In curvature-scaling gravity,  $c = dx/dt \propto a_{\mathcal{R}}^{-1/2} = |\mathcal{R}|^{1/4}$  due to the anisotropic time scaling  $dt \propto a_{\mathcal{R}}^{3/2}$  (whereas  $dx \propto a_{\mathcal{R}}$ ) mentioned above. (See Section 5.3.) Two important points to note:
  - (a) This result strictly protects causality: in each local region,  $c$  is the maximum speed for all objects in the region. No objects can surpass light at any point on the manifold. Superluminality is strictly forbidden. Furthermore, in each region, the Lorentz symmetry holds, with null-geodesics strictly separating timelike trajectories from spacelike trajectories. Causality is protected both locally and globally.
  - (b) The speed of light is not explicitly a function of spacetime coordinates; rather, it is intrinsically a function of the prevailing Ricci scalar. As such, it is permissible for the Lorentz symmetry to hold locally around every point on the manifold.

5. We corrected another deep misperception that the all-embracing utility of  $\hbar$  and  $c$  necessarily meant a universality in their value at all locations. Requirement (I) that physical laws retain their forms in different spacetime pocket forces their parameters – viz.  $\hbar$  and  $c$  – to be functions of the Ricci scalar <sup>35</sup>. In each local region,  $\hbar$  and  $c$  continue to govern over the well-established physics that take place within the region:  $\hbar$  as the fundamental parameter measuring the strength of quantum effects (such as the  $SU(3) \times SU(2) \times U(1)$  model, the phonon and specific heat in solids, the quantum Hall effect, or the nuclear shell model) and  $c$  as the fundamental parameter in the Lorentz symmetry (being the common value of light speed regardless of the direction of the light beam and/or observer.) Yet these parameters are pegged onto the Ricci scalar  $\mathcal{R}$  and thus adapt to the prevailing value of  $\mathcal{R}$  from one region to the next on the manifold, per the scaling rules (57) and (58):  $\hbar \propto |\mathcal{R}|^{-1/4}$ ,  $c \propto |\mathcal{R}|^{1/4}$ , due to the anisotropic time scaling. The all-embracing principles – the causality principle, the relativity principle, Lorentz invariance, the Michelson-Morley finding, the equivalence principle, and the general covariance principle – are fully respected. (See Section 5.2.)

The equivalence principle localizes physical laws (viz. special relativity and quantum laws) to be valid in each pocket of the spacetime manifold. The equivalence principle, in our approach, further localizes the parameters of physical laws to be valid in each pocket only, with the prevailing value of the Ricci scalar directly appoints their values. The parameters that depend on the Ricci scalar are the Planck constant, the speed of light, the length scale and the oscillatory rate of physical processes.

The objectives of our theory are a new construction of the spacetime manifold and the Lagrangian of gravity coupled with matter. We applied the theory to a variety of problems regarding the foundation of relativity and gravity, as well as astrophysics and cosmology:

- We addressed the misplaced fear of violation of causality and Michelson-Morley experimental result. Being local Lorentz invariant, null geodesics remain null geodesics in every coordinate system. Timelike paths and spacelike paths do not mix. At any given point, the speed of light is the upper limit for all objects. Superluminality is strictly forbidden; as such, causality is strictly preserved. (See Section 5.3.)
- In using only dimensionless ratios of lengths with the Ricci length as denominator, gravity arises from matter in an organic, natural, and unique manner. There is only one unified term for the action, viz.  $\mathcal{S}_{CSG} = \int d^4x \mathcal{R}^2 \sqrt{g} \mathcal{L}_m$  which incorporates both matter and gravitation field. There is no “free” gravitation field that lives on its own, detached from matter. Also, the cosmological constant is absent in  $\mathcal{S}_{CSG}$ . (See Section 7.)
- In vacuo, our action  $\mathcal{S}_{CSG}$  above reduces to  $\mathcal{S}_{vacuo} = \int d^4x \sqrt{g} \mathcal{R}^2$  in resemblance of the  $\mathcal{R}^2$  theory. It is a well-posed Cauchy problem with the Cauchy data conveniently consisting of  $g_{\mu\nu}$ ,  $\mathcal{R}$  and their first-order time-derivatives  $\partial_0 g_{\mu\nu}$ ,  $\partial_0 \mathcal{R}$ . (See Section 3.)
- The action  $\mathcal{S}_{vacuo}$  appears to be tractable for static spherically symmetric setup. We provided an explicit solution and established its connection with a solution that Mannheim-Kazanas found in conformal gravity. Their solution and ours possess a new term, corresponding to a linear gravitational potential  $\gamma r$  which acts on the larger scale beyond the solar system. Based on this potential term, Mannheim has endeavored a phenomenological theory to explain the galactic rotation curves without resorting to dark matter. Curvature-scaling gravity thus potentially inherits and strengthens Mannheim’s theory. (See Section 4.)
- We then provided another static spherically symmetric solution which possesses a non-constant Ricci scalar. Close to a mass source, the Ricci scalar diverges at its event horizon, leading to an unbounded growth of  $c$  and a diminishment of  $\hbar$  as one approaches the event horizon. This is a prediction from our theory. The diminishment of  $\hbar$  and quantum effects at the event horizon could alter the radiative behavior of Schwarzschild-type black holes. (See Section 6.)
- For cosmology, we showed that the hypothetical dark energy is nothing but the effects of the adaptation of light speed to the varying Ricci scalar (and, equivalently, to the cosmic scale factor  $a$ .) The standard Friedmann model neglects this important feature, thereby producing flawed redshift formulae and resulting in theoretical predictions irreconcilable with observational data. As a result, standard cosmology had to invent ad hoc solutions – dark energy, inflationary expansion, and accelerating expansion – to reconcile the data with the Friedmann model. All the perceived problems and difficulties in cosmology are non-existent, however; they arose from the deep misperception that  $c$  needs be universal on the spacetime manifold (see Point 4 on the preceding page). By virtue of the equivalence principle, the Lorentz symmetry and its parameter – viz. the speed of light  $c$  – only need be valid locally in each pocket of spacetime.

Starting only from the provision – derivable solely from Requirements (I) and (II) – that  $c$  adapt to the cosmic scale factor per  $c \propto a^{-1/2}$  (58), we provided a unified solution to all 7 most pressing problems simultaneously: (i) the new interpretation of Type 1a supernovae, (ii) the age problem, (iii) the horizon problem, (iv) the flatness problem, (v) Dicke’s instability problem (or the oldness problem), (vi) the budgetary shortfall problem, and (vii) the cosmological

<sup>35</sup> For our comment regarding a line of favorite yet false critique, see footnote 13 on page 17.

constant problem. Our parsimonious solution eliminates the need for the fudge agent of dark energy and the artificial mechanisms of accelerating expansion and inflationary expansion. We corrected the upward bias in the estimation of the Hubble constant (with the corrected value being  $H_0 \approx 37.4$ ) which helps revise the universe's age to 17.4 Glys and reduce the matter budget to 28% of what previously thought. Our solution avoids all fine-tunings and is model-independent. The cosmological constant and all other cosmic coincidences are avoided. (See Sections 8, 9.)

With the misperceived role of causality and the Michelson-Morley experiment corrected, the data of Type Ia supernovae should be interpreted as concrete evidence in support of the variation of  $c$  as function of the cosmic scale factor, rather than an accelerating cosmic expansion. Also, upon the reinterpretation of observational cosmology in the light of curvature-scaling gravity (which supports the notion of  $c$ -variability), the needs for dark energy and inflationary expansion disappear.

Regarding the quantization of gravity, there are desirable properties of  $\mathcal{R}^2$  gravity [14] which could stay valid for curvature-scaling gravity as well, at least in vacuo, in which case curvature-scaling gravity coincides with  $\mathcal{R}^2$  gravity:

- Renormalizability of  $\mathcal{R}^2$  gravity [34]. One nice feature of a renormalizable Lagrangian is that the Lagrangian retains its original form upon the renormalization procedure.
- As a member of the  $f(\mathcal{R})$  class of theories, it is free of ghosts. Thus unitarity is respected [66, 67].
- Regarding Ostrogradsky's instability, in [63] Woodard shows how the  $f(\mathcal{R})$  theories, being degenerate, manage to avoid this fatal instability.
- $\mathcal{R}^2$  gravity avoids Dolgov-Kawasaki's instability [65] since  $f''(\mathcal{R}) = 2 > 0$ .
- $\mathcal{R}^2$  gravity has a well-posed Cauchy problem [12, 13]. The Cauchy data of  $\mathcal{R}^2$  gravity consist of  $g_{\mu\nu}$ ,  $\mathcal{R}$  and their first-order time-derivatives  $\partial_0 g_{\mu\nu}$ ,  $\partial_0 \mathcal{R}$ .

We must however note that in curvature-scaling gravity, the (fixed) Planck scales – Planck energy, Planck length, Planck time – lose their meaning. Since length scale is dependent on the Ricci scalar ( $\hbar$  and  $c$  being dependent on  $\mathcal{R}$  at the location they are active), the concept of a fixed Planck length is no longer meaningful. Likewise, Planck energy and Planck length also lose their meaning. Rather, these quantities are defined locally at each point on the manifold and they are allowed to vary on the manifold. This is another conceptual departure from the standard quantization paradigm. In addition, it is a foremost task to address the conceptual question – which this author leaves unanswered – if the curvature-scaling gravity theory is to be the correct description of gravity at classical level: “What is the actual meaning behind the quantization of spacetime given that  $\hbar$  is curvature-dependent?”<sup>36</sup>

In summary, our theory is an alternative construction of the spacetime manifold: the manifold is a patchwork of local pockets of spacetime, each satisfying special relativity and adopting a local scale. The fundamental constants  $\hbar$  and  $c$  continue to control the physics in each individual local region, yet they must adapt to the prevailing value of the Ricci scalar. (Note that they are not auxiliary fields that live on the manifold.) We preserve all of Einstein's vision and insights in relativity and gravity while at the same time deepening the role of curvature. The Ricci scalar actively partakes in the dynamics of physical processes (by setting the scale for them), thus entailing a small step closer toward a Machian spirit which posits that physics – in this case, the parameters  $\hbar$  and  $c$  – at each point is determined by the distribution of matter in the universe as a whole.

Lastly, our theory is primarily a theory of spacetime and gravitation. It is: (a) Not a theory of time anisotropy<sup>37</sup> or  $c$ -variation per se<sup>38</sup>; (b) Not a theory of cosmology alone. It was not devised to solely resolve problems in cosmology but

<sup>36</sup> An interesting excursion by Mannheim (see Section 4 of [8]) posits a quantization of gravity though its coupling with the quantized matter fields. His thoughts could be of relevance here since in our approach gravity directly arises from matter fields via the replacement of lengths with their dimensionless ratios denominated by the Ricci length. As soon as the matter fields are quantized, would this replacement procedure automatically bring forth a quantized gravitational field too?

<sup>37</sup> In  $2 + 1$  dimensions, time and space would scale similarly,  $dt \propto dx \propto a\mathcal{R}$ . Yet our Requirements (I) and (II) mentioned in the preceding page would continue to hold. (As an aside note, due to the said isotropy,  $c$  and  $\hbar$  would be constant everywhere in  $2 + 1$  dimensions.)

<sup>38</sup> The variability of  $c$  and  $\hbar$  is not an input but instead a by-product of the theory, logically resulting from the dependence of local scale on the Ricci curvature. It is of secondary importance and is a logical consequence of the promotion of the Ricci scalar as the scale-setter for physical processes. In our approach, the variability of  $c$  and  $\hbar$  arises via their endogenous dependency on the Ricci scalar. These parameters do not have a dynamics on their own merit, apart from the dynamics of  $\mathcal{R}$ . Our theory employs no terms such as  $\partial_t c$  or  $\vec{\nabla} \hbar$  which would otherwise inflict structural damages to existing physical laws – a major challenge that plagued the efforts of [52, 53] to build a workable account for variable light speed. For example, in those attempts,  $c$  was allowed to acquire a dynamics governed by some additional quantum mechanical field. It is neither warranted nor necessary to do so in our approach.

We adopt the relativist's view: it is not necessary to design a mechanistic model to account for  $c$ - and  $\hbar$ -variabilities. Here is an apt analogy: prior to the advance of special relativity, the Lorentz-Fitzgerald length contraction and the Lorentz transformation had been devised to account for the Michelson-Morley experiment. Yet Lorentz attributed the length contraction to some yet-to-discover mechanism, such as the interaction between the electrons in the moving ruler and the moving clock with an ether. In contrast, Einstein considered these effects a matter of principle – the relativity principle and the postulate of constant  $c$  – both of which together dictate the form of the Lorentz transformation (and of the length contraction). Indeed Einstein insisted on using the term Relativitätsprinzip (i.e., a descriptive approach) before finally adopting Relativitätstheorie (i.e., an explanatory approach), a term coined by Planck (e.g., see [69]). We follow Einstein's pioneer footstep in this regard.



to address foundational issues of gravitation; (c) Not a theory of  $\mathcal{R}^2$  gravity, although its Lagrangian in vacuo shares a resemblance to the that of  $\mathcal{R}^2$  gravity (the latter not enabling the variations of the local scale and of  $c$  and  $\hbar$ ); (d) Not related to Brans-Dicke or dilaton theories.

Considering the wide-ranging natural outcomes that our new construction of spacetime achieves based on such parsimonious Requirements (I) and (II), we believe it represents an important measure toward the extrapolation of our solar system wisdom into the realm of universe as a whole and the interface with the quantum rules.

Although any scientific endeavor will ultimately be judged by its success or failure to account for physical reality, our theory of curvature-scaling gravity was decidedly originated from a philosophical inspiration: “What sets the size for things around us?”<sup>39</sup>. We were motivated to assign the scale-setter role to the scalar curvature. In so doing, we strengthen the position of the curvature within Einstein’s grand structure of space and time while preserving intact the inner beauty of his insights. Our undertaking is not unlike the conceptual path Einstein adopted in the early stage of his development of general relativity; he was initially guided by the equivalence principle and, to a certain extent, Mach’s principle. These threads of philosophical guidance inspired him to take a leap of faith – that gravity is a manifestation of wrapped spacetime – to arrive at the general covariance principle – that physical laws be expressed in tensorial forms. Only in the later years starting 1912-13 did he – in a frantic race against Hilbert, Einstein the Purist ever converting into Einstein the Pragmatist – abandon the philosophy-inspiring route and adopt a practical approach toward his final gravitational field equations which were mostly aimed at recovering Newtonian gravity in the weak-field limit.

Philosophical enquiries can have encompassing power and scope; their ramification can be far-reaching. It was remarkable – to this author when a young student – that Einstein’s imagination of a hapless falling man eventually offered an explanation for Mercury’s perihelion precession. Throughout this report, we continue the conceptual route that Einstein the Purist departed some time in 1912-13, extend his all-embracing philosophical insights – by ascribing the curvature a new privileged status – and hope to advance his quest for a better understanding of space and time and gravity.

## Acknowledgments

I first and foremost wish to thank Travis W. Fisher for his critical insights proved valuable in the progress of this work. Philip Mannheim helpfully clarified important aspects of his theory of galactic rotation curves. I also like to thank Richard Shurtleff, Soebur Razzaque, Alexandr Yelnikov, V. Parameswaran Nair, Tuan A. Tran, Andrei Pokotilov, Antonio Alfonso-Faus for their constructive feedbacks. Ngoc-Khanh Tran was of crucial help in providing reference material for this work.

---

<sup>39</sup> To the curious-minded reader who, together with the Earth, are in a free fall toward the Sun, the Moon, and all other heavenly bodies, it is tantalizing to inquire what otherwise sets the size of his desk, his office, his own body, as well as the rate of his wristwatch and his own heartbeats. Conventional wisdom would have the omnipotent Planck constant as the scale-setter since quantum mechanics should decide the dimension and vibration rate of the atoms that constitute physical things. We propose an alternative view: it is spacetime itself – in particular, the Ricci scalar – that appoints the value of the Planck constant which in turn sets the size and rate for physical things via quantum mechanics.

## A Derivation of the scaling rule for time duration

Consider the Schrödinger equation of the Hydrogen atom in  $d + 1$  dimensions:

$$i\hbar \frac{\partial}{\partial t} \Psi = -\frac{\hbar^2}{2m_e} \nabla^2 \Psi - \frac{e^2}{r^{d-2}} \Psi \quad (189)$$

Expressing the coordinate differentials in term of their dimensionless:

$$\begin{cases} d\vec{x} &= a_{\mathcal{R}} d\vec{\tilde{x}} \\ dt &= a_{\mathcal{R}}^{\eta} d\tilde{t} \end{cases} \quad (190)$$

in which  $dt$  could acquire a different scale factor than  $dx$  does (hence, the exponent  $\eta$ ). The Schrödinger equation becomes:

$$i \frac{\hbar}{a_{\mathcal{R}}^{\eta}} \frac{\partial}{\partial \tilde{t}} \Psi = -\frac{\hbar^2}{2m_e a_{\mathcal{R}}^2} \tilde{\nabla}^2 \Psi - \frac{e^2}{a_{\mathcal{R}}^{d-2} \tilde{r}^{d-2}} \Psi \quad (191)$$

or

$$i \frac{\hbar}{a_{\mathcal{R}}^{\eta+2-d}} \frac{\partial}{\partial \tilde{t}} \Psi = -\frac{\hbar^2}{2m_e a_{\mathcal{R}}^{4-d}} \tilde{\nabla}^2 \Psi - \frac{e^2}{\tilde{r}^{d-2}} \Psi \quad (192)$$

which requires that

$$\begin{cases} \hbar^2 &\propto a_{\mathcal{R}}^{4-d} \\ \hbar &\propto a_{\mathcal{R}}^{\eta+2-d} \end{cases} \quad (193)$$

and thus

$$\eta = \frac{d}{2}.$$

In  $3 + 1$  dimensions,

$$\eta = \frac{3}{2},$$

the anisotropy in scaling of time and space thus emerges. The anisotropy essentially means that whilst the observer's ruler scales linearly with the Ricci length  $a_{\mathcal{R}}$ , the vibrating rate of his clocks scale as  $a_{\mathcal{R}}^{3/2}$ . In  $2 + 1$  dimensions, time duration and length are isotropic in their scaling behaviors.

### Verification of the scaling rules for QED:

In  $3 + 1$  dimensions, the scaling rules for physical quantities and constants are:

$$\begin{cases} d\vec{x} &\rightarrow a_{\mathcal{R}} d\vec{x} \\ dt &\rightarrow a_{\mathcal{R}}^{3/2} dt \end{cases} \quad (194)$$

and

$$\begin{cases} \vec{p} &\rightarrow a_{\mathcal{R}}^{-1/2} \vec{p} \\ E &\rightarrow a_{\mathcal{R}}^{-1} E \end{cases} \quad (195)$$

whereas

$$\begin{cases} \hbar &\rightarrow a_{\mathcal{R}}^{1/2} \hbar \\ c &\rightarrow a_{\mathcal{R}}^{-1/2} c \\ m &\rightarrow m \\ e &\rightarrow e \\ \alpha &\rightarrow \alpha \\ G &\rightarrow G \end{cases} \quad (196)$$

with  $\vec{x}$ ,  $t$ ,  $\vec{p}$ ,  $E$ ,  $\hbar$ ,  $c$ ,  $m$ ,  $e$ ,  $\alpha$ ,  $G$  being coordinates, time, momentum, energy, the Planck constant, the speed of light, the electron mass, the elementary electric charge, the fine coupling constant, and the gravitational constant respectively. In addition, the electromagnetic potential  $A^{\mu} = (\varphi, \vec{A})$  and fields  $\vec{E} = -\nabla\varphi - \frac{1}{c}\partial_t\vec{A}$ ,  $\vec{B} = \nabla \times \vec{A}$  and fermion field  $\psi$  transform as:

$$\begin{cases} \varphi &\rightarrow a_{\mathcal{R}}^{-1} \varphi \\ \vec{A} &\rightarrow a_{\mathcal{R}}^{-1} \vec{A} \\ \vec{E} &\rightarrow a_{\mathcal{R}}^{-2} \vec{E} \\ \vec{B} &\rightarrow a_{\mathcal{R}}^{-2} \vec{B} \\ \psi &\rightarrow a_{\mathcal{R}}^{-3/2} \psi \end{cases} \quad (197)$$

It is straightforward to verify that the above scaling rules leave the action of QED (with  $x^\mu \triangleq (ct, \vec{x})$ ):

$$\begin{aligned} \mathcal{S} &= \int d^4x \mathcal{L} \\ \mathcal{L} &= \psi^\dagger \left( i\hbar \frac{\partial}{\partial t} + ic\vec{\alpha} \cdot \hbar \vec{\nabla} - mc^2 \beta \right) \psi + e\psi^\dagger \left( -\varphi + \vec{\alpha} \cdot \vec{A} \right) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned} \quad (198)$$

unchanged since  $\mathcal{L} \propto a_{\mathcal{R}}^{-4}$  self-evidently. As such, the (relativistic) Dirac equation and Maxwell equations are unchanged.

### The scaling of energy:

The scaling rules also transform all forms of energy in the same fashion,  $E \propto a_{\mathcal{R}}^{-1}$ . For example:

- The electromagnetic field energy  $\frac{1}{2} \int d\vec{x} (\vec{E}^2 + \vec{B}^2) \propto a_{\mathcal{R}}^3 (a_{\mathcal{R}}^{-2})^2 = a_{\mathcal{R}}^{-1}$  since  $\vec{E} \propto a_{\mathcal{R}}^{-2}$  and  $\vec{B} \propto a_{\mathcal{R}}^{-2}$ .
- Photon energy  $E = \hbar\nu \propto a_{\mathcal{R}}^{1/2} a_{\mathcal{R}}^{-3/2} = a_{\mathcal{R}}^{-1}$ .
- Energy of a massive object:  $E = mc^2 \propto a_{\mathcal{R}}^{-1}$ .
- Energy levels of the hydrogen atom  $E_{jn} = -\frac{m_e e^4}{2\hbar^2 n^2} \left[ 1 + \frac{\alpha^2}{n^2} \left( \frac{n}{j+\frac{1}{2}} - \frac{3}{4} \right) \right] \propto a_{\mathcal{R}}^{-1}$ .

The requirement that all forms of energy obey a universal scaling rule is important. Let us stress that our consideration is unconventional in contrast to the standard cosmological paradigm which discriminates radiation from matter and the ‘‘Doppler theft’’ is said to only affect photons but leave nonrelativistic matter untouched. In our theory, all forms of energy obey a common scaling rule,  $E \propto a_{\mathcal{R}}^{-1}$ .

## B Buchdahl's treatment of $\mathcal{R}^2$ gravity revisited

In this section, we shall revisit Buchdahl's formulation of  $\mathcal{R}^2$ -field equation in vacuo. In so doing, we simplify some parts of his derivation and correct a number of typos in his paper. Toward to end is a new representation of his results. We also illustrate the 4 degrees of freedom for spherical coordinate. Following Buchdahl's notation, the metric in spherical coordinate is written in the form:

$$\begin{aligned} ds^2 &= -e^\nu (dx^0)^2 + e^\lambda dr^2 + e^\mu d\Omega^2 \\ d\Omega^2 &= d\theta^2 + \sin^2 \theta d\varphi^2 \end{aligned} \quad (199)$$

The relevant components of the Ricci and metric tensors and Christoffel symbols are:

$$\begin{cases} \mathcal{R}_{tt} e^{\lambda-\nu} &= \frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'\lambda'}{4} + \frac{\nu'\mu'}{2} \\ -\mathcal{R}_{\theta\theta} e^{\lambda-\mu} &= -e^{\lambda-\mu} + \frac{\mu''}{2} + \frac{\mu'^2}{2} + \frac{\nu'\mu'}{4} - \frac{\lambda'\mu'}{4} \\ -\mathcal{R}_{rr} &= \frac{\nu''}{2} + \frac{\nu'^2}{4} + \mu'' + \frac{\mu'^2}{2} - \frac{\nu'\lambda'}{4} - \frac{\lambda'\mu'}{2} \end{cases} \quad (200)$$

$$\begin{cases} g_{tt} e^{-\nu} &= -1 \\ g_{\theta\theta} e^{-\mu} &= 1 \\ g_{rr} e^{-\lambda} &= 1 \end{cases} \quad (201)$$

$$\begin{cases} \Gamma_{tt}^r e^{\lambda-\nu} &= \frac{\nu'}{2} \\ \Gamma_{\theta\theta}^r e^{\lambda-\mu} &= -\frac{\mu'}{2} \\ \Gamma_{rr}^r &= \frac{\lambda'}{2} \end{cases} \quad (202)$$

$$-\mathcal{R}e^\lambda = -2e^{\lambda-\mu} + \nu'' + \frac{\nu'^2}{2} + 2\mu'' + \frac{3\mu'^2}{2} - \frac{\nu'\lambda'}{2} + \nu'\mu' - \lambda'\mu' \quad (203)$$

Furthermore, the Jacobian:

$$\sqrt{g} \triangleq \sqrt{-\det g} = e^{\frac{\nu}{2} + \frac{\lambda}{2} + \mu} \sin^2 \theta \quad (204)$$

$$\sqrt{g} g^{rr} = e^{\frac{\nu}{2} - \frac{\lambda}{2} + \mu} \sin^2 \theta \quad (205)$$

The three functions  $\nu$ ,  $\lambda$ ,  $\mu$  are subject to an arbitrary coordinate transform. Buchdahl chooses that:

$$\mu = \frac{1}{2}(\lambda - \nu) \quad (206)$$

so that  $\sqrt{g} g^{rr} = \sin^2 \theta$ . As such, the equation

$$\square \mathcal{R} = 0 \quad (207)$$

or its equivalent form

$$(\sqrt{g} g^{rr} \mathcal{R}')' = 0 \quad (208)$$

is simplified to

$$\mathcal{R}'' = 0 \quad (209)$$

or

$$\mathcal{R} = \Lambda + kr \quad (210)$$

where  $\Lambda$  and  $k$  are constants of integration. Note that, in Buchdahl's coordinate, the limit of spatial infinity corresponds to  $r \rightarrow 0$ . Thus  $\Lambda$  corresponds to the large-distance curvature (de Sitter parameter).

With Buchdahl's choice, the relevant components become:

$$\begin{cases} \mathcal{R}_{tt} &= \frac{\nu''}{2} e^{\nu-\lambda} \\ \mathcal{R}_{\theta\theta} &= 1 + e^{-\frac{\nu}{2}-\frac{\lambda}{2}} \left( \frac{\nu''}{4} - \frac{\lambda''}{4} \right) \\ \mathcal{R}_{rr} &= -\frac{\lambda''}{2} + \frac{\lambda'^2}{8} - \frac{3\nu'^2}{8} + \frac{\nu'\lambda'}{4} \\ \mathcal{R} &= 2e^{\frac{\nu}{2}-\frac{\lambda}{2}} - e^{-\lambda} \left( \lambda'' - \frac{\lambda'^2}{8} - \frac{\lambda'\nu'}{4} + \frac{3\nu'^2}{8} \right) \end{cases} \quad (211)$$

The field equations in  $\mathcal{R}^2$  Lagrangian are:

$$\begin{cases} \mathcal{R}_{tt} - \frac{1}{4}g_{tt}\mathcal{R} &= -\Gamma_{tt}^r \frac{\mathcal{R}'}{\mathcal{R}} \\ \mathcal{R}_{\theta\theta} - \frac{1}{4}g_{\theta\theta}\mathcal{R} &= -\Gamma_{\theta\theta}^r \frac{\mathcal{R}'}{\mathcal{R}} \\ \mathcal{R}_{rr} - \frac{1}{4}g_{rr}\mathcal{R} &= -\Gamma_{rr}^r \frac{\mathcal{R}'}{\mathcal{R}} + \frac{\mathcal{R}''}{\mathcal{R}} \end{cases} \quad (212)$$

The  $tt$ -equation:

$$\frac{\nu''}{2} e^{\nu-\lambda} + \frac{1}{4} e^{\nu} (\Lambda + kr) = -\frac{\nu'}{2} e^{\nu-\lambda} \frac{k}{\Lambda + kr} \quad (213)$$

$$\nu'' + \frac{k}{\Lambda + kr} \nu' + \frac{1}{2} (\Lambda + kr) e^{\lambda} = 0 \quad (214)$$

The  $\theta\theta$ -equation:

$$1 + e^{-\frac{\nu}{2}-\frac{\lambda}{2}} \left( \frac{\nu''}{4} - \frac{\lambda''}{4} \right) - \frac{1}{4} e^{\frac{\lambda}{2}-\frac{\nu}{2}} (\Lambda + kr) = \left( \frac{\lambda'}{4} - \frac{\nu'}{4} \right) e^{-\frac{\nu}{2}-\frac{\lambda}{2}} \frac{k}{\Lambda + kr} \quad (215)$$

$$\lambda'' - \nu'' + \frac{k}{\Lambda + kr} (\lambda' - \nu') + (\Lambda + kr) e^{\lambda} = 4e^{\frac{\nu}{2}+\frac{\lambda}{2}} \quad (216)$$

which, combined with (214), becomes:

$$\lambda'' + \frac{k}{\Lambda + kr} \lambda' + \frac{3}{2} (\Lambda + kr) e^{\lambda} = 4e^{\frac{\nu}{2}+\frac{\lambda}{2}} \quad (217)$$

The  $rr$ -equation:

$$-\frac{\lambda''}{2} + \frac{\lambda'^2}{8} - \frac{3\nu'^2}{8} + \frac{\nu'\lambda'}{4} - \frac{1}{4} e^{\lambda} (\Lambda + kr) = -\frac{\lambda'}{2} \frac{k}{\Lambda + kr} \quad (218)$$

$$\lambda'' - \frac{k}{\Lambda + kr} \lambda' + \frac{1}{2} (\Lambda + kr) e^{\lambda} - \frac{\lambda'^2}{4} + \frac{3\nu'^2}{4} - \frac{\nu'\lambda'}{2} = 0 \quad (219)$$

From the definition of  $\mathcal{R}$ :

$$2e^{\frac{\nu}{2}-\frac{\lambda}{2}} - e^{-\lambda} \left( \lambda'' - \frac{\lambda'^2}{8} + \frac{3\nu'^2}{8} - \frac{\nu'\lambda'}{4} \right) = \Lambda + kr \quad (220)$$

$$\lambda'' - \frac{\lambda'^2}{8} + \frac{3\nu'^2}{8} - \frac{\nu'\lambda'}{4} + (\Lambda + kr) e^{\lambda} = 2e^{\frac{\nu}{2}+\frac{\lambda}{2}} \quad (221)$$

Note that the  $\mathcal{R}$ -equation is the average of the  $\theta\theta$ - and  $rr$ -equations. Now, eliminating  $\lambda''$  from the  $\theta\theta$ - and  $\mathcal{R}$ -equations, we get:

$$2e^{\frac{\nu}{2} + \frac{\lambda}{2}} - \frac{k}{\Lambda + kr} \lambda' - \frac{1}{2} (\Lambda + kr) e^\lambda - \frac{\lambda'^2}{8} + \frac{3\nu'^2}{8} - \frac{\nu' \lambda'}{4} = 0 \quad (222)$$

In summary, the two equations determine  $\nu$  and  $\lambda$  as functions of  $r$ . The last equation is not an additional “constraint” but simply a “conservation law” among  $\nu$ ,  $\lambda$ ,  $\nu'$ ,  $\lambda'$  at any given  $r$ .

Next, we make a coordinate change, which is slightly different from Buchdahl's:

$$\Lambda + kr = \pm \Lambda e^{kz} \quad (223)$$

$$\begin{aligned} \frac{d}{dr} &= \frac{dz}{dr} \frac{d}{dz} = \pm \frac{e^{-kz}}{\Lambda} \frac{d}{dz} \\ \frac{d^2}{dr^2} &= \frac{dz}{dr} \frac{d}{dz} \left( \pm \frac{e^{-kz}}{\Lambda} \frac{d}{dz} \right) = \pm \frac{e^{-kz}}{\Lambda} \left( \mp \frac{k e^{-kz}}{\Lambda} \frac{d}{dz} \pm \frac{e^{-kz}}{\Lambda} \frac{d^2}{dz^2} \right) = \frac{e^{-2kz}}{\Lambda^2} \left( \frac{d^2}{dz^2} - k \frac{d}{dz} \right) \end{aligned}$$

The three equations become:

$$\begin{cases} \frac{e^{-2kz}}{\Lambda^2} (\nu_{zz} - k\nu_z) + \frac{k e^{-2kz}}{\Lambda^2} \nu_z \pm \frac{\Lambda}{2} e^{kz+\lambda} &= 0 \\ \frac{e^{-2kz}}{\Lambda^2} (\lambda_{zz} - k\lambda_z) + \frac{k e^{-2kz}}{\Lambda^2} \lambda_z \pm \frac{3\Lambda}{2} e^{kz+\lambda} &= 4e^{\frac{\nu}{2} + \frac{\lambda}{2}} \\ \frac{k e^{-2kz}}{\Lambda^2} \lambda_z \pm \frac{\Lambda}{2} e^{kz+\lambda} + \frac{e^{-2kz}}{8\Lambda^2} \lambda_z^2 - \frac{3e^{-2kz}}{8\Lambda^2} \nu_z^2 + \frac{e^{-2kz}}{4\Lambda^2} \nu_z \lambda_z &= 2e^{\frac{\nu}{2} + \frac{\lambda}{2}} \end{cases} \quad (224)$$

or

$$\begin{cases} \nu_{zz} \pm \frac{\Lambda^3}{2} e^{3kz+\lambda} &= 0 \\ \lambda_{zz} \pm \frac{3\Lambda^3}{2} e^{3kz+\lambda} &= 4\Lambda^2 e^{2kz+\frac{\nu}{2}+\frac{\lambda}{2}} \\ \lambda_z^2 - 3\nu_z^2 + 2\nu_z \lambda_z + 8k\lambda_z \pm 4\Lambda^3 e^{3kz+\lambda} &= 16\Lambda^2 e^{2kz+\frac{\nu}{2}+\frac{\lambda}{2}} \end{cases} \quad (225)$$

Further defining

$$\begin{cases} \nu &= -u + v - kz - \ln \frac{\Lambda}{4} \\ \lambda &= 3u + v - 3kz - 3 \ln \frac{\Lambda}{4} \\ \mu &= \frac{\lambda}{2} - \frac{\nu}{2} = 2u - kz - \ln \frac{\Lambda}{4} \end{cases} \quad (226)$$

we get

$$\begin{cases} u_{zz} &= 16e^u (1 \mp e^{2u}) e^v \\ v_{zz} &= 16e^u (1 \mp 3e^{2u}) e^v \\ u_z v_z &= 16e^u (1 \mp e^{2u}) e^v + \frac{3k^2}{4} \end{cases} \quad (227)$$

It is also easy to check that upon taking derivative w.r.t  $z$ , the last equation is consistent with the two former equations. Therefore, we could interpret – *à la* Buchdahl – the first two equations as “equations of motion” (in “time”  $z$ ) and the last equation as a “conservation law” in which  $k$  is a “constant of motion”. The parameter  $k$  also plays another role: it sets the degree of deviation of the curvature away from the constant  $\Lambda$ . A non-zero value of  $k$  means a static metric with non-constant Ricci scalar.

Buchdahl next exploited some clever analogy to Hamiltonian dynamics to simplify these equations. However, with the benefit of hindsight, we find a shortcut which we present in what follows:

Upon differentiating the last equation w.r.t.  $z$ :

$$u_{zz} v_z + u_z v_{zz} = 16 (e^u \mp 3e^{3u}) e^v u_z + 16 (e^u \mp e^{3u}) e^v v_z \quad (228)$$

and utilizing the first equation, we re-obtain the second equation. Thus, we can ignore the second equation from now on.

Define  $q$  as a function of  $u$ :

$$q = u_z \quad (229)$$

Also, viewing  $v$  as a function of  $u$ :

$$u_{zz} = q_z = q_u u_z = q_u q \quad (230)$$

$$v_z = v_u u_z = v_u q \quad (231)$$

The first and last equations become:

$$qq_u = 16e^u (1 \mp e^{2u}) e^v \quad (232)$$

$$q^2 v_u = 16e^u (1 \mp e^{2u}) e^v + \frac{3k^2}{4} \quad (233)$$

Now, setting

$$u = \ln x \quad (234)$$

$$q_u = \frac{q_x}{u_x} = x q_x \quad (235)$$

$$v_u = \frac{v_x}{u_x} = x v_x \quad (236)$$

we thus get:

$$q q_x = 16 (1 \mp x^2) e^v \quad (237)$$

$$q^2 v_x = 16 (1 \mp x^2) e^v + \frac{3k^2}{4x} = q q_x + \frac{3k^2}{4x} \quad (238)$$

Differentiating the first equation w.r.t.  $x$ :

$$\begin{aligned} q_x^2 + q q_{xx} &= 16 (1 \mp x^2) e^v v_x \mp 32 x e^v \\ &= q q_x v_x \mp \frac{2x q q_x}{1 \mp x^2} \\ &= q_x^2 + \frac{3k^2 q_x}{4x q} \mp \frac{2x q q_x}{1 \mp x^2} \end{aligned} \quad (239)$$

$$q_{xx} \pm \frac{2x}{1 \mp x^2} q_x = \frac{3k^2}{4x q^2} q_x \quad (240)$$

$$q_{xx} - \frac{2x}{x^2 \mp 1} q_x = \frac{3k^2}{4x q^2} q_x \quad (241)$$

$$\partial_x \left( \frac{q_x}{x^2 \mp 1} \right) = \frac{3k^2}{4x q} \left( \frac{q_x}{x^2 \mp 1} \right) \quad (242)$$

This is the equation that Buchdahl obtained.

### Our representation:

Define a new function  $p$  as function of  $x$ :

$$p = \frac{q_x}{1 \mp x^2} \quad (243)$$

We thus have:

$$q_x = (1 \mp x^2) p \quad (244)$$

$$p_x = \frac{3k^2}{4x} \frac{p}{q} \quad (245)$$

In terms of  $x$ :

$$e^u = x \quad (246)$$

$$e^v = \frac{q q_x}{16 (1 \mp x^2)} = \frac{q p}{16} \quad (247)$$

$$e^\nu = e^{-u+v-kz-\ln \frac{\Lambda}{4}} = \frac{4}{\Lambda e^{kz}} \frac{q p}{16 x} \quad (248)$$

$$e^\lambda = e^{3u+v-3kz-3 \ln \frac{\Lambda}{4}} = \frac{64}{\Lambda^3 e^{3kz}} \frac{x^3 q p}{16} \quad (249)$$

$$e^\mu = e^{2u-kz-\ln \frac{\Lambda}{4}} = \frac{4}{\Lambda e^{kz}} x^2 \quad (250)$$

With  $\mathcal{R} = \Lambda + k r = \pm \Lambda e^{kz}$ :

$$dr = \pm \Lambda e^{kz} dz$$

and we also know that

$$q = u_z = \frac{x_z}{x} = \frac{1}{x} \frac{dx}{dz} \quad (251)$$

$$dz = \frac{dx}{x q} \quad (252)$$

$$dr = \pm \Lambda e^{kz} \frac{1}{x q} dx \quad (253)$$

the metric becomes:

$$\begin{aligned}
 ds^2 &= -e^\nu (dx^0)^2 + e^\lambda dr^2 + e^\mu d\Omega^2 \\
 &= -\frac{4}{\Lambda e^{kz}} \frac{qp}{16x} (dx^0)^2 + \frac{64}{\Lambda^3 e^{3kz}} \frac{x^3 qp}{16} \left( \Lambda^2 e^{2kz} \frac{1}{x^2 q^2} dx^2 \right) + \frac{4}{\Lambda e^{kz}} x^2 d\Omega^2 \\
 &= \frac{4}{\Lambda e^{kz}} \left\{ \frac{p}{4} \left[ -\frac{q}{4x} (dx^0)^2 + \frac{4x}{q} dx^2 \right] + x^2 d\Omega^2 \right\}
 \end{aligned} \tag{254}$$

In summary, the metric is:

$$ds^2 = \frac{4}{\mathcal{R}(x)} \left\{ \frac{p(x)}{4} \left[ -\frac{q(x)}{4x} (dx^0)^2 + \frac{4x}{q(x)} dx^2 \right] + x^2 d\Omega^2 \right\} \tag{255}$$

in which

$$\begin{cases} \mathcal{R}(x) &= \pm \Lambda e^{k \int \frac{dx}{xq(x)}} \\ q_x &= (1 \mp x^2) p \\ p_x &= \frac{3k^2}{4x} \frac{p}{q} \end{cases} \tag{256}$$

The 4 parameters:  $k$  (the anomalous curvature),  $\Lambda$  (the large-scale curvature),  $p(x_0)$  and  $q(x_0)$  at some  $x_0$  of convenience. Note that there is a constant of integration in  $\int \frac{dx}{xq(x)}$  but it can be absorbed into  $\Lambda$ .

## C An explicit solution for spherically symmetric case

Let us find a solution in the form:

$$ds^2 = e^\alpha \left[ -\Psi (dx^0)^2 + \frac{dr^2}{\Psi} + r^2 d\Omega^2 \right] \tag{257}$$

The relevant quantities are computed

$$\begin{cases} \frac{\mathcal{R}_{tt}}{\Psi} &= \left( \frac{\alpha''}{2} + \frac{\alpha'^2}{2} + \frac{\alpha'}{r} \right) \Psi + \left( \alpha' + \frac{1}{r} \right) \Psi' + \frac{\Psi''}{2} \\ -\frac{\mathcal{R}_{\theta\theta}}{r^2} &= -\frac{1}{r^2} + \left( \frac{\alpha''}{2} + \frac{\alpha'^2}{2} + \frac{2\alpha'}{r} + \frac{1}{r^2} \right) \Psi + \left( \frac{\alpha'}{2} + \frac{1}{r} \right) \Psi' \\ \mathcal{A} \triangleq -\mathcal{R}e^\alpha &= -\frac{2}{r^2} + \left( 3\alpha'' + \frac{3\alpha'^2}{2} + \frac{6\alpha'}{r} + \frac{2}{r^2} \right) \Psi + \left( 3\alpha' + \frac{4}{r} \right) \Psi' + \Psi'' \end{cases} \tag{258}$$

$$\begin{cases} g_{tt} &= -e^\alpha \Psi \\ g_{\theta\theta} &= e^\alpha r^2 \\ \frac{\Gamma_{tt}^r}{\Psi} &= \frac{\alpha' \Psi}{2} + \frac{\Psi'}{2} \\ \frac{\Gamma_{\theta\theta}^r}{r^2} &= -\left( \frac{\alpha'}{2} + \frac{1}{r} \right) \Psi \end{cases} \tag{259}$$

The  $tt$ - and  $\theta\theta$ - field equations

$$\left( \mathcal{R}_{tt} - \frac{1}{4} g_{tt} \mathcal{R} \right) \mathcal{R} = -\Gamma_{tt}^r \mathcal{R}' \tag{260}$$

$$\left( \mathcal{R}_{\theta\theta} - \frac{1}{4} g_{\theta\theta} \mathcal{R} \right) \mathcal{R} = -\Gamma_{\theta\theta}^r \mathcal{R}' \tag{261}$$

can be recast as

$$\left( \frac{\mathcal{R}_{tt}}{\Psi} + \frac{1}{4} \frac{g_{tt} e^{-\alpha}}{\Psi} (-\mathcal{R}e^\alpha) \right) (-\mathcal{R}e^\alpha) = \frac{\Gamma_{tt}^r}{\Psi} (\mathcal{R}' e^\alpha) \tag{262}$$

$$\left( -\frac{\mathcal{R}_{\theta\theta}}{r^2} - \frac{1}{4} \frac{g_{\theta\theta} e^{-\alpha}}{r^2} (-\mathcal{R}e^\alpha) \right) (-\mathcal{R}e^\alpha) = -\frac{\Gamma_{\theta\theta}^r}{r^2} (\mathcal{R}' e^\alpha) \tag{263}$$

With

$$\mathcal{A}' = -\mathcal{R}' e^\alpha - \alpha' \mathcal{R} e^\alpha \tag{264}$$

$$\mathcal{R}' e^\alpha = -\mathcal{A}' + \alpha' \mathcal{A} \tag{265}$$

we thus obtain three equations for the two unknowns  $\alpha$ ,  $\Psi$  and the auxiliary  $\mathcal{A}$ :

$$\left[ \left( \frac{\alpha''}{2} + \frac{\alpha'^2}{2} + \frac{\alpha'}{r} \right) \Psi + \left( \alpha' + \frac{1}{r} \right) \Psi' + \frac{\Psi''}{2} - \frac{\mathcal{A}}{4} \right] \mathcal{A} = \left( \frac{\alpha' \Psi}{2} + \frac{\Psi'}{2} \right) (-\mathcal{A}' + \alpha' \mathcal{A}) \quad (266)$$

$$\left[ -\frac{1}{r^2} + \left( \frac{\alpha''}{2} + \frac{\alpha'^2}{2} + \frac{2\alpha'}{r} + \frac{1}{r^2} \right) \Psi + \left( \frac{\alpha'}{2} + \frac{1}{r} \right) \Psi' - \frac{\mathcal{A}}{4} \right] \mathcal{A} = \left( \frac{\alpha'}{2} + \frac{1}{r} \right) \Psi (-\mathcal{A}' + \alpha' \mathcal{A}) \quad (267)$$

$$-\frac{2}{r^2} + \left( 3\alpha'' + \frac{3\alpha'^2}{2} + \frac{6\alpha'}{r} + \frac{2}{r^2} \right) \Psi + \left( 3\alpha' + \frac{4}{r} \right) \Psi' + \Psi'' = \mathcal{A} \quad (268)$$

We can simplify these equation further by directly varying the action:

$$\mathcal{S} = \int d^4x \sqrt{g} \mathcal{R}^2 = \int dx^0 dr d\theta d\varphi \sin^2 \theta e^{2\alpha} r^2 \mathcal{R}^2 = 4\pi \int dx^0 dr r^2 \mathcal{A}^2 \quad (269)$$

in which

$$\sqrt{g} = e^{2\alpha} r^2 \sin^2 \theta \quad (270)$$

$$\sqrt{g} g^{rr} = e^\alpha \Psi r^2 \sin^2 \theta \quad (271)$$

The first equation of motion:

$$\left( \frac{\partial (r^2 \mathcal{A}^2)}{\partial \alpha''} \right)'' - \left( \frac{\partial (r^2 \mathcal{A}^2)}{\partial \alpha'} \right)' + \frac{\partial (r^2 \mathcal{A}^2)}{\partial \alpha} = 0 \quad (272)$$

$$\left( \frac{\partial (r^2 \mathcal{A}^2)}{\partial \alpha''} \right)' - \frac{\partial (r^2 \mathcal{A}^2)}{\partial \alpha'} = c$$

$$(r^2 \mathcal{A}) \left[ \left( \frac{\partial \mathcal{A}}{\partial \alpha''} \right)' - \frac{\partial \mathcal{A}}{\partial \alpha'} \right] + (r^2 \mathcal{A})' \frac{\partial \mathcal{A}}{\partial \alpha''} = c$$

$$\left( \frac{\partial \mathcal{A}}{\partial \alpha''} \right)' - \frac{\partial \mathcal{A}}{\partial \alpha'} = 3 \left[ \Psi' - \left( \alpha' \Psi + \frac{2\Psi}{r} + \Psi' \right) \right] = -3 \left( \alpha' + \frac{2}{r} \right) \Psi$$

$$\frac{\partial \mathcal{A}}{\partial \alpha''} = 3\Psi$$

$$(r^2 \mathcal{A})' = r^2 \mathcal{A}' + 2r \mathcal{A} = r^2 \left( \mathcal{A}' + \frac{2}{r} \mathcal{A} \right)$$

$$-\left( \alpha' + \frac{2}{r} \right) r^2 \Psi \mathcal{A} + r^2 \Psi \left( \mathcal{A}' + \frac{2}{r} \mathcal{A} \right) = c$$

$$(\mathcal{A}' - \alpha' \mathcal{A}) r^2 \Psi = c$$

$$\mathcal{R}' e^\alpha r^2 \Psi = c \quad (273)$$

which is equivalent to

$$\square \mathcal{R} = 0.$$

Indeed

$$(\sqrt{g} g^{rr} \mathcal{R}')' = 0$$

or

$$(e^\alpha \Psi r^2 \mathcal{R}')' = 0.$$

The second equation of motion:

$$\left( \frac{\partial (r^2 \mathcal{A}^2)}{\partial \Psi''} \right)'' - \left( \frac{\partial (r^2 \mathcal{A}^2)}{\partial \Psi'} \right)' + \frac{\partial (r^2 \mathcal{A}^2)}{\partial \Psi} = 0 \quad (274)$$

$$(r^2 \mathcal{A}) \left[ \left( \frac{\partial \mathcal{A}}{\partial \Psi''} \right)'' - \left( \frac{\partial \mathcal{A}}{\partial \Psi'} \right)' + \frac{\partial \mathcal{A}}{\partial \Psi} \right] + (r^2 \mathcal{A})' \left[ 2 \left( \frac{\partial \mathcal{A}}{\partial \Psi''} \right)' - \frac{\partial \mathcal{A}}{\partial \Psi'} \right] + (r^2 \mathcal{A})'' \frac{\partial \mathcal{A}}{\partial \Psi''} = 0$$



$$\begin{aligned}
\left(\frac{\partial \mathcal{A}}{\partial \Psi''}\right)'' - \left(\frac{\partial \mathcal{A}}{\partial \Psi'}\right)' + \frac{\partial \mathcal{A}}{\partial \Psi} &= -\left(3\alpha'' - \frac{4}{r^2}\right) + \left(3\alpha'' + \frac{3\alpha'^2}{2} + \frac{6\alpha'}{r} + \frac{2}{r^2}\right) = \frac{3\alpha'^2}{2} + \frac{6\alpha'}{r} + \frac{6}{r^2} \\
2\left(\frac{\partial \mathcal{A}}{\partial \Psi''}\right)' - \frac{\partial \mathcal{A}}{\partial \Psi'} &= -\left(3\alpha' + \frac{4}{r}\right) \\
\frac{\partial \mathcal{A}}{\partial \Psi''} &= 1 \\
(r^2 \mathcal{A})' &= r^2 \mathcal{A}' + 2r \mathcal{A} = r^2 \left(\mathcal{A}' + \frac{2}{r} \mathcal{A}\right) \\
(r^2 \mathcal{A})'' &= r^2 \mathcal{A}'' + 4r \mathcal{A}' + 2\mathcal{A} = r^2 \left(\mathcal{A}'' + \frac{4}{r} \mathcal{A}' + \frac{2}{r^2} \mathcal{A}\right) \\
\left(\frac{3\alpha'^2}{2} + \frac{6\alpha'}{r} + \frac{6}{r^2}\right) r^2 \mathcal{A} - \left(3\alpha' + \frac{4}{r}\right) r^2 \left(\mathcal{A}' + \frac{2}{r} \mathcal{A}\right) + r^2 \left(\mathcal{A}'' + \frac{4}{r} \mathcal{A}' + \frac{2}{r^2} \mathcal{A}\right) &= 0 \\
\mathcal{A}'' - 3\alpha' \mathcal{A}' + \frac{3\alpha'^2}{2} \mathcal{A} &= 0 \\
\mathcal{R}'' - \alpha' \mathcal{R}' + \left(\alpha'' - \frac{\alpha'^2}{2}\right) \mathcal{R} &= 0
\end{aligned} \tag{275}$$

Let us find the solution with  $c = 0$ , which implies  $\mathcal{R}' = 0$ . Then

$$\alpha'' - \frac{\alpha'^2}{2} = 0 \tag{276}$$

$$\alpha = -2 \ln(1 + ar) \tag{277}$$

$$e^\alpha = \frac{1}{(1 + ar)^2} \tag{278}$$

$$\alpha' = -\frac{2a}{1 + ar} \tag{279}$$

and

$$\mathcal{A}' = \alpha' \mathcal{A} \tag{280}$$

$$\mathcal{A} = -\mathcal{R}_0 e^\alpha = -\frac{\mathcal{R}_0}{(1 + ar)^2} \tag{281}$$

Plug this into Eq. (267):

$$-\frac{1}{r^2} + \left(\frac{\alpha''}{2} + \frac{\alpha'^2}{2} + \frac{2\alpha'}{r} + \frac{1}{r^2}\right) \Psi + \left(\frac{\alpha'}{2} + \frac{1}{r}\right) \Psi' = \frac{\mathcal{A}}{4} = -\frac{\mathcal{R}_0}{4} e^\alpha \tag{282}$$

With

$$\frac{\alpha''}{2} + \frac{\alpha'^2}{2} + \frac{2\alpha'}{r} + \frac{1}{r^2} = \frac{3a^2}{(1 + ar)^2} - \frac{4a}{r(1 + ar)} + \frac{1}{r^2} = \frac{1 - 2ar}{r^2(1 + ar)^2} \tag{283}$$

and

$$\frac{\alpha'}{2} + \frac{1}{r} = -\frac{a}{1 + ar} + \frac{1}{r} = \frac{1}{r(1 + ar)} \tag{284}$$

$$r(1 + ar) \Psi' + (1 - 2ar) \Psi - (1 + ar)^2 + \frac{\mathcal{R}_0}{4} r^2 = 0 \tag{285}$$

or

$$(1 + ar) (r\Psi)' - 3a(r\Psi) - (1 + ar)^2 + \frac{\mathcal{R}_0}{4} r^2 = 0 \tag{286}$$

Choose

$$r\Psi = br - r_s - \Lambda r^3 + \gamma r^2 \tag{287}$$

the equation successively becomes

$$(1 + ar) (b - 3\Lambda r^2 + 2\gamma r) - 3a(br - r_s - \Lambda r^3 + \gamma r^2) - 1 - 2ar - a^2 r^2 + \frac{\mathcal{R}_0}{4} r^2 = 0 \tag{288}$$

$$b - 3\Lambda r^2 + 2\gamma r + abr - 3a\Lambda r^3 + 2a\gamma r^2 - 3abr + 3ar_s + 3a\Lambda r^3 - 3a\gamma r^2 - 1 - 2ar - a^2 r^2 + \frac{\mathcal{R}_0}{4} r^2 = 0 \quad (289)$$

$$(b + 3ar_s - 1) + 2(\gamma - ab - a)r + \left(-3\Lambda - a\gamma - a^2 + \frac{\mathcal{R}_0}{4}\right) r^2 = 0 \quad (290)$$

which solves for

$$\begin{cases} b &= 1 - 3ar_s \\ \gamma &= ab + a = a(2 - 3ar_s) \\ \Lambda &= \frac{\mathcal{R}_0}{12} - \frac{a\gamma + a^2}{3} = \frac{\mathcal{R}_0}{12} + a^2(ar_s - 1) \end{cases} \quad (291)$$

It is straightforward to verify that  $\Psi$  and  $\alpha$  with  $b$ ,  $\gamma$ ,  $\Lambda$  given above also satisfy Eqs. (266) and (268). In summary, one spherically symmetric solution to  $\mathcal{R}^2$  Lagrangian is:

### Three representations:

#### 1. Representation I:

$$\begin{cases} \Psi &= (1 - 3ar_s) - \frac{r_s}{r} - \Lambda r^2 + a(2 - 3ar_s)r \\ e^\alpha &= (1 + ar)^{-2} \\ \Lambda &= \frac{\mathcal{R}_0}{12} + a^2(ar_s - 1) \end{cases} \quad (292)$$

#### 2. Representation II (analogous to Mannheim-Kazanas):

If we define

$$\begin{cases} ar_s &\triangleq \beta\gamma \\ a(2 - 3ar_s) &\triangleq \gamma \end{cases} \quad (293)$$

which leads to

$$\begin{cases} a &= \frac{\gamma}{2 - 3ar_s} = \frac{\gamma}{2 - 3\beta\gamma} \\ r_s &= \frac{\beta\gamma}{a} = \beta(2 - 3\beta\gamma) \end{cases} \quad (294)$$

we thus get:

$$\begin{cases} \Psi &= (1 - 3\beta\gamma) - \frac{\beta(2 - 3\beta\gamma)}{r} - \Lambda r^2 + \gamma r \\ e^\alpha &= \left(1 + \frac{\gamma}{2 - 3\beta\gamma} r\right)^{-2} \end{cases} \quad (295)$$

#### 3. Representation III:

Define a dimensionless parameter  $\kappa$ :

$$\kappa \triangleq -3\gamma r_s \quad (296)$$

then we successively get

$$\begin{aligned} -\frac{\kappa}{3} &= \gamma r_s = 2(ar_s) - 3(ar_s)^2 \\ ar_s &= \frac{1 - \sqrt{1 + \kappa}}{3} \\ b &= 1 - 3ar_s = \sqrt{1 + \kappa} \\ a^2(ar_s - 1) &= -\frac{1}{27r_s^2} (1 - \sqrt{1 + \kappa})^2 (2 + \sqrt{1 + \kappa}) \end{aligned}$$

in which the sign of the square root is chosen such that for small  $\kappa$ ,  $a \simeq \kappa$  and  $b \approx 1$ . We thus get:

$$\begin{cases} \Psi &= \sqrt{1 + \kappa} - \frac{r_s}{r} - \Lambda r^2 - \frac{\kappa}{3r_s} r \\ e^\alpha &= \left(1 + \frac{1 - \sqrt{1 + \kappa}}{3r_s} r\right)^{-2} \\ \Lambda &= \frac{\mathcal{R}_0}{12} - \frac{1}{27r_s^2} (1 - \sqrt{1 + \kappa})^2 (2 + \sqrt{1 + \kappa}) \end{cases} \quad (297)$$

## D A perturbative solution to the vacuo field equation: The anomalous curvature

The metric is written in the following form with two unknown functions  $\alpha$  and  $\Psi$ :

$$ds^2 = e^\alpha \left[ -\Psi (dx^0)^2 + \frac{dr^2}{\Psi} + r^2 d\Omega^2 \right] \quad (298)$$

Taking a cue from our previous solution, we shall find the two unknown functions in the form:

$$\Psi = \Psi_0 + \gamma \Psi_1 + \mathcal{O}(\gamma^2) \quad (299)$$

$$\alpha = \gamma \Phi_1 + \mathcal{O}(\gamma^2) \quad (300)$$

in which

$$\Psi_0 = 1 - \frac{r_s}{r} - \Lambda r^2 \quad (301)$$

That is to say, we shall find the uniformly convergent solutions with  $\gamma$  as a perturbative parameter. The parameter  $\gamma$  shall play a role similar to that of the Mannheim-Kazanas parameter in our solution in Appendix C.

$$\begin{aligned} \frac{\mathcal{R}_{tt}}{\Psi} &= \left( \frac{\alpha''}{2} + \frac{\alpha'^2}{2} + \frac{\alpha'}{r} \right) \Psi + \frac{\Psi''}{2} + \left( \alpha' + \frac{1}{r} \right) \Psi' \\ &= \gamma \left( \frac{\Phi_1''}{2} + \frac{\Phi_1'}{r} \right) \Psi_0 + \frac{\Psi_0''}{2} + \gamma \frac{\Psi_1''}{2} + \frac{\Psi_0'}{r} + \gamma \left( \frac{\Psi_1'}{r} + \Phi_1' \Psi_0' \right) + \mathcal{O}(\gamma^2) \\ &= \left( \frac{\Psi_0''}{2} + \frac{\Psi_0'}{r} \right) + \gamma \left( \frac{\Phi_1'' \Psi_0}{2} + \frac{\Phi_1' \Psi_0}{r} + \frac{\Psi_1''}{2} + \frac{\Psi_1'}{r} + \Phi_1' \Psi_0' \right) + \mathcal{O}(\gamma^2) \\ &= -3\Lambda + \gamma \left( \frac{\Phi_1'' \Psi_0}{2} + \frac{\Phi_1' \Psi_0}{r} + \frac{\Psi_1''}{2} + \frac{\Psi_1'}{r} + \Phi_1' \Psi_0' \right) + \mathcal{O}(\gamma^2) \end{aligned} \quad (302)$$

$$\begin{aligned} -\frac{\mathcal{R}_{\theta\theta}}{r^2} &= -\frac{1}{r^2} + \left( \frac{\alpha''}{2} + \frac{\alpha'^2}{2} + \frac{2\alpha'}{r} + \frac{1}{r^2} \right) \Psi + \left( \frac{\alpha'}{2} + \frac{1}{r} \right) \Psi' \\ &= -\frac{1}{r^2} + \frac{\Psi_0}{r^2} + \gamma \left( \frac{\Phi_1'' \Psi_0}{2} + \frac{2\Phi_1' \Psi_0}{r} + \frac{\Psi_1}{r^2} \right) + \frac{\Psi_0'}{r} + \gamma \left( \frac{\Psi_1'}{r} + \frac{\Phi_1' \Psi_0'}{2} \right) + \mathcal{O}(\gamma^2) \\ &= \left( -\frac{1}{r^2} + \frac{\Psi_0}{r^2} + \frac{\Psi_0'}{r} \right) + \gamma \left( \frac{\Phi_1'' \Psi_0}{2} + \frac{2\Phi_1' \Psi_0}{r} + \frac{\Psi_1}{r^2} + \frac{\Psi_1'}{r} + \frac{\Phi_1' \Psi_0'}{2} \right) + \mathcal{O}(\gamma^2) \\ &= -3\Lambda + \gamma \left( \frac{\Phi_1'' \Psi_0}{2} + \frac{2\Phi_1' \Psi_0}{r} + \frac{\Psi_1}{r^2} + \frac{\Psi_1'}{r} + \frac{\Phi_1' \Psi_0'}{2} \right) + \mathcal{O}(\gamma^2) \end{aligned} \quad (303)$$

$$\begin{aligned} -\mathcal{R}e^\alpha &= -\frac{2}{r^2} + \left( 3\alpha'' + \frac{3\alpha'^2}{2} + \frac{6\alpha'}{r} + \frac{2}{r^2} \right) \Psi + \left( 3\alpha' + \frac{4}{r} \right) \Psi' + \Psi'' \\ &= -\frac{2}{r^2} + \frac{2\Psi_0}{r^2} + \gamma \left( 3\Phi_1'' \Psi_0 + \frac{6\Phi_1' \Psi_0}{r} + \frac{2\Psi_1}{r^2} \right) + \frac{4\Psi_0'}{r} + \gamma \left( \frac{4\Psi_1'}{r} + 3\Phi_1' \Psi_0' \right) + \Psi_0'' + \gamma \Psi_1'' + \mathcal{O}(\gamma^2) \\ &= \left( -\frac{2}{r^2} + \frac{2\Psi_0}{r^2} + \frac{4\Psi_0'}{r} + \Psi_0'' \right) + \gamma \left( 3\Phi_1'' \Psi_0 + \frac{6\Phi_1' \Psi_0}{r} + \frac{2\Psi_1}{r^2} + \frac{4\Psi_1'}{r} + 3\Phi_1' \Psi_0' + \Psi_1'' \right) + \mathcal{O}(\gamma^2) \\ &= -12\Lambda + \gamma \left( 3\Phi_1'' \Psi_0 + \frac{6\Phi_1' \Psi_0}{r} + \frac{2\Psi_1}{r^2} + \frac{4\Psi_1'}{r} + 3\Phi_1' \Psi_0' + \Psi_1'' \right) + \mathcal{O}(\gamma^2) \end{aligned} \quad (304)$$

$$\begin{aligned} \mathcal{R}'e^\alpha &= (\mathcal{R}e^\alpha)' - \alpha' (\mathcal{R}e^\alpha) \\ &= -\gamma \left( 3\Phi_1'' \Psi_0 + \frac{6\Phi_1' \Psi_0}{r} + \frac{2\Psi_1}{r^2} + \frac{4\Psi_1'}{r} + 3\Phi_1' \Psi_0' + \Psi_1'' \right)' - 12\Lambda \gamma \Phi_1' + \mathcal{O}(\gamma^2) \end{aligned} \quad (305)$$

$$\frac{(\mathcal{R}'e^\alpha)}{(-\mathcal{R}e^\alpha)} = \frac{\gamma}{12\Lambda} \left[ \left( 3\Phi_1'' \Psi_0 + \frac{6\Phi_1' \Psi_0}{r} + \frac{2\Psi_1}{r^2} + \frac{4\Psi_1'}{r} + 3\Phi_1' \Psi_0' + \Psi_1'' \right)' + 12\Lambda \Phi_1' \right] + \mathcal{O}(\gamma^2) \quad (306)$$

$$\begin{cases} \frac{g_{tt}e^{-\alpha}}{\Psi} = -1 \\ \frac{g_{\theta\theta}e^\alpha}{r^2} = 1 \\ \frac{\Gamma_{tt}^r}{\Psi} = \frac{\alpha'\Psi}{2} + \frac{\Psi'}{2} = \frac{\Psi_0'}{2} + \mathcal{O}(\gamma) \\ -\frac{\Gamma_{\theta\theta}^r}{r^2} = \left( \frac{\alpha'}{2} + \frac{1}{r} \right) \Psi = \frac{\Psi_0}{r} + \mathcal{O}(\gamma) \end{cases} \quad (307)$$

The  $tt$ - and  $\theta\theta$ -field equations:

$$\left(\mathcal{R}_{tt} - \frac{1}{4}g_{tt}\mathcal{R}\right)\mathcal{R} = -\Gamma_{tt}^r\mathcal{R}' \quad (308)$$

$$\left(\mathcal{R}_{\theta\theta} - \frac{1}{4}g_{\theta\theta}\mathcal{R}\right)\mathcal{R} = -\Gamma_{\theta\theta}^r\mathcal{R}' \quad (309)$$

can be cast as

$$\frac{\mathcal{R}_{tt}}{\Psi} + \frac{1}{4}\frac{g_{tt}e^{-\alpha}}{\Psi}(-\mathcal{R}e^\alpha) = \frac{\Gamma_{tt}^r(\mathcal{R}'e^\alpha)}{\Psi(-\mathcal{R}e^\alpha)} \quad (310)$$

$$-\frac{\mathcal{R}_{\theta\theta}}{r^2} - \frac{1}{4}\frac{g_{\theta\theta}e^{-\alpha}}{r^2}(-\mathcal{R}e^\alpha) = -\frac{\Gamma_{\theta\theta}^r(\mathcal{R}'e^\alpha)}{r^2(-\mathcal{R}e^\alpha)} \quad (311)$$

Up to the first-order in  $\gamma$ , the equations for the two unknowns  $\Psi_1$  and  $\Phi_1$  are:

$$\begin{aligned} \frac{\Psi_0\Phi_1''}{2} + \Psi_0'\Phi_1' + \frac{\Psi_0\Phi_1'}{r} + \frac{\Psi_1''}{2} + \frac{\Psi_1'}{r} &= \frac{1}{4}\left(3\Psi_0\Phi_1'' + 3\Psi_0'\Phi_1' + 6\frac{\Psi_0\Phi_1'}{r} + \Psi_1'' + 4\frac{\Psi_1'}{r} + 2\frac{\Psi_1}{r^2}\right) \\ &= \frac{\Psi_0'}{24\Lambda}\left[\left(3\Psi_0\Phi_1'' + 3\Psi_0'\Phi_1' + 6\frac{\Psi_0\Phi_1'}{r} + \Psi_1'' + 4\frac{\Psi_1'}{r} + 2\frac{\Psi_1}{r^2}\right)' + 12\Lambda\Phi_1'\right] \end{aligned} \quad (312)$$

$$\begin{aligned} \frac{\Psi_0\Phi_1''}{2} + \frac{\Psi_0'\Phi_1'}{2} + 2\frac{\Psi_0\Phi_1'}{r} + \frac{\Psi_1'}{r} + \frac{\Psi_1}{r^2} &= \frac{1}{4}\left(3\Psi_0\Phi_1'' + 3\Psi_0'\Phi_1' + 6\frac{\Psi_0\Phi_1'}{r} + \Psi_1'' + 4\frac{\Psi_1'}{r} + 2\frac{\Psi_1}{r^2}\right) \\ &= \frac{\Psi_0}{12\Lambda r}\left[\left(3\Psi_0\Phi_1'' + 3\Psi_0'\Phi_1' + 6\frac{\Psi_0\Phi_1'}{r} + \Psi_1'' + 4\frac{\Psi_1'}{r} + 2\frac{\Psi_1}{r^2}\right)' + 12\Lambda\Phi_1'\right] \end{aligned} \quad (313)$$

First, define:

$$\mathcal{E} \triangleq \frac{\Psi_0\Phi_1''}{2} + \frac{\Psi_0'\Phi_1'}{2} + \frac{\Psi_0\Phi_1'}{r} \quad (314)$$

which can simplified further

$$\begin{aligned} \mathcal{E} &= \frac{1}{2}(\Psi_0\Phi_1')' + \frac{1}{r}(\Psi_0\Phi_1') \\ &= \frac{1}{2r^2}(r^2\Psi_0\Phi_1')' \end{aligned} \quad (315)$$

Next, define:

$$\mathcal{F} \triangleq \frac{1}{4}\left(3\Psi_0\Phi_1'' + 3\Psi_0'\Phi_1' + 6\frac{\Psi_0\Phi_1'}{r} + \Psi_1'' + 4\frac{\Psi_1'}{r} + 2\frac{\Psi_1}{r^2}\right) \quad (316)$$

which can be simplified further

$$\mathcal{F} = \frac{3}{2}\mathcal{E} + \frac{1}{4}\left(\Psi_1'' + 4\frac{\Psi_1'}{r} + 2\frac{\Psi_1}{r^2}\right) \quad (317)$$

and also

$$\begin{aligned} \mathcal{F} &= \frac{1}{4}\left(\frac{3}{r^2}(r^2\Psi_0\Phi_1')' + \frac{1}{r^2}(r^2\Psi_1)''\right) \\ &= \frac{1}{4r^2}\left(3r^2\Psi_0\Phi_1' + (r^2\Psi_1)'\right)' \end{aligned} \quad (318)$$

Eqs. (312) and (313) become:

$$\left(\frac{\Psi_0\Phi_1''}{2} + \Psi_0'\Phi_1' + \frac{\Psi_0\Phi_1'}{r} + \frac{\Psi_1''}{2} + \frac{\Psi_1'}{r}\right) - \frac{\Psi_0'\Phi_1'}{2} = \mathcal{F} + \frac{\Psi_0'}{6\Lambda}\mathcal{F}' \quad (319)$$

$$\left(\frac{\Psi_0\Phi_1''}{2} + \frac{\Psi_0'\Phi_1'}{2} + 2\frac{\Psi_0\Phi_1'}{r} + \frac{\Psi_1'}{r} + \frac{\Psi_1}{r^2}\right) - \frac{\Psi_0\Phi_1'}{r} = \mathcal{F} + \frac{\Psi_0}{3\Lambda r}\mathcal{F}' \quad (320)$$

The left-hand-sides of Eqs. (319) and (320) are

$$\text{LHS of (319)} = \mathcal{E} + \frac{\Psi_1''}{2} + \frac{\Psi_1'}{r} \quad (321)$$

$$\text{LHS of (320)} = \mathcal{E} + \frac{\Psi_1}{r^2} + \frac{\Psi_1'}{r} \quad (322)$$

Inverting (317)

$$\mathcal{E} = \frac{2}{3}\mathcal{F} - \frac{1}{6} \left( \Psi_1'' + 4\frac{\Psi_1'}{r} + 2\frac{\Psi_1}{r^2} \right) \quad (323)$$

the LHS's become

$$\text{LHS of (319)} = \frac{2}{3}\mathcal{F} + \frac{\Psi_1''}{3} + \frac{\Psi_1'}{3r} - \frac{\Psi_1}{3r^2} \quad (324)$$

$$\text{LHS of (320)} = \frac{2}{3}\mathcal{F} - \frac{\Psi_1''}{6} + \frac{\Psi_1'}{3r} + 2\frac{\Psi_1}{3r^2} \quad (325)$$

Eqs. (319) and (320) can be simplified to be

$$\frac{\Psi_0'}{2\Lambda}\mathcal{F}' + \mathcal{F} = \Psi_1'' + \frac{\Psi_1'}{r} - \frac{\Psi_1}{r^2} \quad (326)$$

$$\frac{\Psi_0}{\Lambda r}\mathcal{F}' + \mathcal{F} = -\frac{\Psi_1''}{2} + \frac{\Psi_1'}{r} + 2\frac{\Psi_1}{r^2} \quad (327)$$

which, as algebraic equations, can be solved for  $\mathcal{F}$  and  $\mathcal{F}'$  separately:

$$\mathcal{F}' = 3\Lambda \frac{\Psi_1'' - 2\frac{\Psi_1}{r^2}}{\Psi_0' - 2\frac{\Psi_0}{r}} = 3\Lambda \frac{\Psi_1'' - 2\frac{\Psi_1}{r^2}}{r^2 (r^{-2}\Psi_0)'} \quad (328)$$

$$\mathcal{F} = -\frac{\Psi_1''}{2} + \frac{\Psi_1'}{r} + 2\frac{\Psi_1}{r^2} - \frac{\Psi_0}{\Lambda r}\mathcal{F}' \quad (329)$$

Next, define:

$$\mathcal{G} \triangleq \Psi_1'' - 2\frac{\Psi_1}{r^2} \quad (330)$$

which leads to

$$\mathcal{G} = r \left( r^{-2} (r\Psi_1)' \right)' \quad (331)$$

$$\mathcal{G}' = \Psi_1''' - 2\frac{\Psi_1'}{r^2} + 4\frac{\Psi_1}{r^3} \quad (332)$$

We then have, from Eqs. (328) and (329):

$$\mathcal{F}' = 3\Lambda \frac{\mathcal{G}}{r^2 (r^{-2}\Psi_0)'} \quad (333)$$

$$\mathcal{F} = -\frac{\Psi_1''}{2} + \frac{\Psi_1'}{r} + 2\frac{\Psi_1}{r^2} - 3\frac{\Psi_0\mathcal{G}}{r^3 (r^{-2}\Psi_0)'} \quad (334)$$

Differentiating Eq. (334) and invoking (330) and (332):

$$\begin{aligned} \mathcal{F}' &= -\frac{\Psi_1'''}{2} + \frac{\Psi_1''}{r} + \frac{\Psi_1'}{r^2} - 4\frac{\Psi_1}{r^3} - 3\frac{\Psi_0}{r^3 (r^{-2}\Psi_0)'}\mathcal{G}' - 3\left(\frac{\Psi_0}{r^3 (r^{-2}\Psi_0)'}\right)' \mathcal{G} \\ &= -\frac{1}{2}\left(\Psi_1''' - 2\frac{\Psi_1'}{r^2} + 4\frac{\Psi_1}{r^3}\right) + \frac{1}{r}\left(\Psi_1'' - 2\frac{\Psi_1}{r^2}\right) - 3\frac{\Psi_0}{r^3 (r^{-2}\Psi_0)'}\mathcal{G}' - 3\left(\frac{\Psi_0}{r^3 (r^{-2}\Psi_0)'}\right)' \mathcal{G} \\ &= -\frac{1}{2}\mathcal{G}' + \frac{1}{r}\mathcal{G} - 3\frac{\Psi_0}{r^3 (r^{-2}\Psi_0)'}\mathcal{G}' - 3\left(\frac{\Psi_0}{r^3 (r^{-2}\Psi_0)'}\right)' \mathcal{G} \\ &= -\left[3\frac{\Psi_0}{r^3 (r^{-2}\Psi_0)'} + \frac{1}{2}\right]\mathcal{G}' + \left[-3\left(\frac{\Psi_0}{r^3 (r^{-2}\Psi_0)'}\right)' + \frac{1}{r}\right]\mathcal{G} \end{aligned} \quad (335)$$

Equating Eqs. (333) and (335):

$$\left[ 3 \frac{\Psi_0}{r^3 (r^{-2}\Psi_0)'} + \frac{1}{2} \right] \mathcal{G}' = \left[ -3 \left( \frac{\Psi_0}{r^3 (r^{-2}\Psi_0)'} \right)' + \frac{1}{r} - 3\Lambda \frac{1}{r^2 (r^{-2}\Psi_0)'} \right] \mathcal{G} \quad (336)$$

The bracketed terms in Eq. (336) are explicitly computed below:

$$\begin{aligned} \Psi_0 &= 1 - \frac{r_s}{r} - \Lambda r^2 \\ r^{-2}\Psi_0 &= \frac{1}{r^2} - \frac{r_s}{r^3} - \Lambda \\ (r^{-2}\Psi_0)' &= -\frac{2}{r^3} + \frac{3r_s}{r^4} = -\frac{2}{r^4} \left( r - \frac{3}{2}r_s \right) \\ \frac{1}{r^2 (r^{-2}\Psi_0)'} &= -\frac{1}{2} \frac{r^2}{r - \frac{3}{2}r_s} \\ \frac{\Psi_0}{r^3 (r^{-2}\Psi_0)'} &= -\frac{1}{2} \frac{r - r_s - \Lambda r^3}{r - \frac{3}{2}r_s} \\ \left( \frac{\Psi_0}{r^3 (r^{-2}\Psi_0)'} \right)' &= -\frac{1}{2} \frac{(1 - 3\Lambda r^2) (r - \frac{3}{2}r_s) - (r - r_s - \Lambda r^3)}{(r - \frac{3}{2}r_s)^2} = \frac{\frac{1}{4}r_s + \Lambda r^3 - \frac{9}{4}r_s \Lambda r^2}{(r - \frac{3}{2}r_s)^2} \end{aligned}$$

and

$$3 \frac{\Psi_0}{r^3 (r^{-2}\Psi_0)'} + \frac{1}{2} = -\frac{3}{2} \frac{r - r_s - \Lambda r^3}{r - \frac{3}{2}r_s} + \frac{1}{2} = \frac{-r + \frac{3}{4}r_s + \frac{3}{2}\Lambda r^3}{r - \frac{3}{2}r_s} \quad (337)$$

and

$$-3 \left( \frac{\Psi_0}{r^3 (r^{-2}\Psi_0)'} \right)' + \frac{1}{r} - 3\Lambda \frac{1}{r^2 (r^{-2}\Psi_0)'} = \frac{-\frac{3}{4}r_s - 3\Lambda r^3 + \frac{27}{4}r_s \Lambda r^2}{(r - \frac{3}{2}r_s)^2} + \frac{1}{r} + \frac{\frac{3}{2}\Lambda r^2}{r - \frac{3}{2}r_s} = \frac{(-r + 3r_s) (-r + \frac{3}{4}r_s + \frac{3}{2}\Lambda r^3)}{r (r - \frac{3}{2}r_s)^2} \quad (338)$$

Plugging (337) and (338) to Eq. (336), we obtain:

$$\frac{\mathcal{G}'}{\mathcal{G}} = \frac{-r + 3r_s}{r (r - \frac{3}{2}r_s)} = \frac{1}{r - \frac{3}{2}r_s} - \frac{2}{r} \quad (339)$$

from which

$$\mathcal{G} = -\frac{a}{r^2} \left( r - \frac{3}{2}r_s \right) \quad (340)$$

with  $a$  being a constant of integration. Combined with (331) we then get

$$\Psi_1 = \frac{a}{2} \left( r - \frac{3}{2}r_s \right) + br^2 + \frac{c}{r} \quad (341)$$

with  $b$  and  $c$  being two additional constants of integration. However,  $a$ ,  $b$  and  $c$  can be absorbed into the definition of  $\Lambda$ ,  $r_s$  and  $\gamma$  respectively in  $\Psi_0$ . We thus set  $a = 2$ ,  $b = c = 0$  and obtain

$$\Psi_1 = r - \frac{3}{2}r_s \quad (342)$$

which, via (334), neatly leads to

$$\begin{aligned} \mathcal{F} &= -\frac{\Psi_1''}{2} + \frac{\Psi_1'}{r} + 2\frac{\Psi_1}{r^2} - 3\frac{\Psi_0}{r^3 (r^{-2}\Psi_0)'} \mathcal{G} \\ &= \frac{1}{r} + \frac{2}{r} - \frac{3r_s}{r^2} + 3 \left( 1 - \frac{r_s}{r} - \Lambda r^2 \right) \frac{1}{r^3 - \frac{2}{r^3} + \frac{3r_s}{r^4}} \frac{2}{r^2} \left( r - \frac{3}{2}r_s \right) \\ &= \frac{3}{r} - \frac{3r_s}{r^2} - 3 \left( 1 - \frac{r_s}{r} - \Lambda r^2 \right) \frac{1}{r} \\ &= 3\Lambda r \end{aligned} \quad (343)$$

Using (318), we successively get

$$\begin{aligned} \left(3r^2\Psi_0\Phi'_1 + (r^2\Psi_1)'\right)' &= 12\Lambda r^3 \\ 3r^2\Psi_0\Phi'_1 + 3r^2 - 3r_s r &= 3\Lambda r^4 + 3\epsilon \\ r^2\Psi_0\Phi'_1 &= -(r^2 - r_s r - \Lambda r^4) + \epsilon \\ \Phi'_1 &= -1 + \frac{\epsilon}{r^2\Psi_0} \end{aligned}$$

or, finally:

$$\Phi_1 = -r + \epsilon \int \frac{dr}{r^2 \left(1 - \frac{r_s}{r} - \Lambda r^2\right)} \quad (344)$$

The Ricci scalar can also be calculated from (304) and (316):

$$\mathcal{R}e^\alpha = 12\Lambda - 4\gamma\mathcal{F} + \mathcal{O}(\gamma^2)$$

or, using (300), (343), and (344):

$$\begin{aligned} \mathcal{R} &= 12\Lambda(1 - \alpha) - 4\gamma\mathcal{F} + \mathcal{O}(\gamma^2) \\ &= 12\Lambda - \gamma(4\mathcal{F} + 12\Lambda\Phi_1) + \mathcal{O}(\gamma^2) \\ &= 12\Lambda \left[1 - \gamma\epsilon \int \frac{dr}{r^2 \left(1 - \frac{r_s}{r} - \Lambda r^2\right)}\right] + \mathcal{O}(\gamma^2) \end{aligned} \quad (345)$$

In perfect agreement with Buchdahl's study in Appendix B, the solution (342, 344) contains 4 parameters: (1)  $\Lambda$ , the large-distance curvature (the de Sitter term); (2)  $r_s$  as a free parameter (the Newton term); (3)  $\gamma$ , specifying the linear term as in Mannheim-Kazanas's potential in conformal gravity (the Mannheim-Kazanas term); and (4)  $\epsilon$ , characterizing the anomalous curvature – it allows the curvature to be non-constant as evident in (345). The parameter  $\epsilon$  is the new feature in our solution. There is a constant of integration in  $\Phi_1$  but it is an overall scale factor.

In summary, the metric is:

$$ds^2 = e^{\gamma \left[ -r + \epsilon \int \frac{dr}{r^2 \left(1 - \frac{r_s}{r} - \Lambda r^2\right)} \right]} \left[ - \left(1 - \frac{r_s}{r} - \Lambda r^2 + \gamma \left(r - \frac{3}{2}r_s\right)\right) (dx^0)^2 + \frac{dr^2}{1 - \frac{r_s}{r} - \Lambda r^2 + \gamma \left(r - \frac{3}{2}r_s\right)} + r^2 d\Omega^2 \right] + \mathcal{O}(\gamma^2) \quad (346)$$

## E The Ricci scalar in the Robertson-Walker metric and the modified Robertson-Walker metric

Let us first consider the closed universe case,  $\kappa = 1$ . The RW metric reads (with  $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ ):

$$ds^2 = c^2 dt^2 - a^2(t) (d\chi^2 + \sin^2\chi d\Omega^2) \quad (347)$$

whereas the modified RW metric reads

$$ds^2 = \left(c_0 \sqrt{\frac{a_0}{a(t)}}\right)^2 dt^2 - a^2(t) (d\chi^2 + \sin^2\chi d\Omega^2) = c_0^2 \frac{a_0}{a(t)} dt^2 - a^2(t) (d\chi^2 + \sin^2\chi d\Omega^2) \quad (348)$$

Both metrics can be commonly recast in term of the conformal time  $\eta$ :

$$ds^2 = a^2(\eta) [d\eta^2 - (d\chi^2 + \sin^2\chi d\Omega^2)] \quad (349)$$

with  $\eta$  being defined as

$$d\eta = \begin{cases} c a^{-1}(t) dt & \text{for RW metric} \\ c_0 a_0^{1/2} a^{-3/2}(t) dt & \text{for modified RW metric} \end{cases} \quad (350)$$

The variables  $(x^0, x^1, x^2, x^3) \triangleq (\eta, \chi, \theta, \phi)$  are of dimensionless unit. The metric tensor is diagonal:

$$g_{00} = a^2(\eta), \quad g_{11} = -a^2(\eta), \quad g_{22} = -a^2(\eta) \sin^2\chi, \quad g_{33} = -a^2(\eta) \sin^2\chi \sin^2\theta. \quad (351)$$

Of the 64 Christoffel symbols  $\Gamma_{\mu\nu}^{\alpha} = \frac{1}{2}g^{\alpha\beta}(\partial_{\mu}g_{\nu\beta} + \partial_{\nu}g_{\mu\beta} - \partial_{\beta}g_{\mu\nu})$ , the 19 non-vanishing components are (with the prime indicating the derivative with respect to  $\eta$ ):

$$\begin{aligned}\Gamma_{00}^0 &= \Gamma_{11}^0 = \Gamma_{01}^1 = \Gamma_{10}^1 = \Gamma_{02}^2 = \Gamma_{20}^2 = \Gamma_{03}^3 = \Gamma_{30}^3 = \frac{a'}{a}, \\ \Gamma_{22}^0 &= \frac{a'}{a} \sin^2 \chi, \quad \Gamma_{33}^0 = \frac{a'}{a} \sin^2 \chi \sin^2 \theta, \quad \Gamma_{22}^1 = -\sin \chi \cos \chi, \quad \Gamma_{33}^1 = -\sin \chi \cos \chi \sin^2 \theta, \\ \Gamma_{12}^2 &= \Gamma_{21}^2 = \Gamma_{13}^3 = \Gamma_{31}^3 = \cot \chi, \quad \Gamma_{33}^2 = -\sin \theta \cos \theta, \quad \Gamma_{23}^3 = \Gamma_{32}^3 = \cot \theta\end{aligned}\quad (352)$$

The Ricci tensor  $\mathcal{R}_{\beta}^{\alpha} = g^{\alpha\gamma}(\partial_{\delta}\Gamma_{\gamma\beta}^{\delta} - \partial_{\beta}\Gamma_{\gamma\delta}^{\delta} + \Gamma_{\gamma\beta}^{\delta}\Gamma_{\delta\sigma}^{\sigma} - \Gamma_{\gamma\delta}^{\sigma}\Gamma_{\beta\sigma}^{\delta})$  is diagonal:

$$\mathcal{R}_0^0 = -\frac{3}{a^2} \left( \frac{a''}{a} - \frac{a'^2}{a^2} \right), \quad \mathcal{R}_k^k = -\frac{1}{a^2} \left( \frac{a''}{a} + \frac{a'^2}{a^2} + 2 \right) \quad (353)$$

The Ricci scalar is

$$\mathcal{R} = \mathcal{R}_{\alpha}^{\alpha} = -\frac{6}{a^2} \left( \frac{a''}{a} + 1 \right) \quad (354)$$

The case of open universe,  $\kappa = -1$ , can be obtained by replacing  $\sin \chi \rightarrow \sinh \chi$  in the modified RW metric. This amounts to replacing  $\chi \rightarrow i\chi$ . The case of flat universe,  $\kappa = 0$ , can be obtained by replacing  $\sin \chi \rightarrow \chi$  in the modified RW metric. The previous calculations sail through with the Ricci scalar being generalized to

$$\mathcal{R} = -\frac{6}{a^2} \left( \frac{a''}{a} + \kappa \right) \quad (355)$$

This result is applicable to both the RW metric and the modified RW metric.

## F Three forms of the time duration paradox

The time duration paradox (also known as the twin paradox in popular science) is probably the most perplexing one in relativity. The most classic version of it started in the early day of special relativity and is usually cast in a fanciful form which involves a pair of identical twins. One twin stays on the Earth while the other travels at high speed to a distant star light years away then makes a U-turn to come back. At their rendezvous decades later, the twin that stays on Earth will have aged more than the other one. The twin paradox has been resolved. There are two other versions of it, though, one in general relativity and one in our theory of curvature-scaling gravity. To bring forth our version, it is necessary that we review the other two for the purpose of comparison.

### 1. The classic and original version, valid for special relativity:

A pair of synchronized clocks, starting at one location  $A$ , are made to trace out two timelike trajectories then brought back together at a later time at  $B$ . For each clock, the time count (i.e., the number of revolutions the clock will have completed, or the number of “beats” the clock will have clicked) between the two events is  $\Delta\tau = \Delta s/c$  with  $\Delta s \triangleq \int_A^B ds$  and  $ds^2 = c^2 dt^2 - d\vec{x}^2$  being the infinitesimal proper distance in the Minkowski metric. If the cumulative proper distances of the two paths are different, say,  $\Delta s_1 > \Delta s_2$ , the clocks will register different time counts,  $\Delta\tau_1 > \Delta\tau_2$ . The metric (whether it is Minkowski or not) is not important; the only requirement is that the two path accumulate different total proper distances. This effect is related to the time dilation effect, observed in the life time of muons which are created in the stratosphere yet could reach the sea level.

### 2. The gravitational redshift, which takes place in general relativity:

This effect was one of the early predictions of the equivalence principle. It has been verified in the Pound-Rebka experiment which measures the redshift of photons free falling in the Earth’s gravitation field. We shall recast this paradox in a form more aligned with the first version. Two synchronized clocks, starting at one location  $A$ , are quickly sent to two different regions, one in a strong gravitational field and the other in a weak field. After a long exposure to the fields, they are quickly brought back together. Again, the time count for each clock is  $\Delta\tau = \Delta s/c$  with  $\Delta s = \int_A^B ds$  as before but  $ds^2 = g_{00} c^2 dt^2 + \dots$  with the dots denotes the remaining 15 terms in the metric. These 15 terms vanish along the two trajectories since the two clocks sit still at their respective regions for the whole long exposure. Take the Schwarzschild metric:  $g_{00} = 1 - \frac{r_s}{r}$  in which  $r_s = \frac{GM}{c^2}$  where  $M$  is the mass of the field source. Therefore, although  $\Delta t_1 = \Delta t_2$ , the cumulative proper distances are different due to the difference in their  $g_{00}$ . Take  $M_1 > M_2$ , then  $g_{00}(1) < g_{00}(2)$  and so,  $\Delta s_1 < \Delta s_2$ . As such, the clocks register different time counts:  $\Delta\tau_1 < \Delta\tau_2$ . In this case, the clock exposed to the stronger gravitational field runs slower (i.e., registers less time counts) than the clock exposed to the weaker field, as expected from the equivalence principle.



### 3. The third version – Time counts in curvature-scaling gravity:

Consider again the two synchronized clocks. Send them to two regions with different Ricci scalars, say,  $\mathcal{R}_1 > \mathcal{R}_2$ , which means their Ricci lengths are related as  $a_{\mathcal{R}}(1) < a_{\mathcal{R}}(2)$ . Note that the time count is no longer  $\Delta\tau = \Delta s/c$  but is the ratio of  $\Delta\tau$  with the oscillatory period of the atoms that make up the clock. With  $a_{\mathcal{R}}(1) < a_{\mathcal{R}}(2)$ , the atoms in Region 1 require less time to vibrate than the atoms in Region 2. The time scaling rule (55) dictates that the period for each vibration scale as  $a_{\mathcal{R}}^{3/2}$ . Therefore, even if the total proper distances of the two paths are deliberately prepared to be equal, Clock 1 registers more time counts than Clock 2:  $\Delta\tilde{\tau}_1 > \Delta\tilde{\tau}_2$ . This effect works somewhat in reverse of the gravitational redshift, since  $\mathcal{R}_1 > \mathcal{R}_2$  would usually require a stronger field for Region 1 than the field in Region 2. We do not expect this effect to be material for the solar system however since the solar system is almost Ricci flat.

## G On the redshift of photons from distant galaxies

Although the Universe has been in the expansion mode since the Big Bang, galaxies are not subject to cosmic expansion. Otherwise, the redshift of light emitted from distant galaxies would not be detectable on Earth because Earth-based apparatus would expand accordingly and thus the traveling photon's wavelength and/or frequency would appear exactly the same as the one on Earth.

In this Appendix, we establish the relationship between redshift parameter  $z$  to the cosmic scale expansion within our curvature-scaling gravity. Note that a new element in our theory is the variability of light speed, in which  $c \propto a^{-1/2}$  with  $a$  being the cosmic scale factor at the location the photon resides.

We model the Milky Way and a distant galaxy as non-expanding objects, devoid of the cosmic expansion. The scale factor for them are set as a fixed value, denoted as  $a_0 = 1$ . Nonetheless, we do allow the space enclosing the galaxies to expand. We even allow the space to have been expanding well before the galaxies were formed. With such allowance, each of the galaxies and the space enclosing it can have different scale factor.

Consider a photon emitted from the distant galaxy at wavelength  $\lambda_0$  and travel at speed  $c_0$ . The outskirts of the distant galaxy had been expanding to a scale factor  $a_1 > 1$ , thus corresponding to a lower light speed  $c_1 = \frac{c_0}{\sqrt{a_1}}$ . Once reaching the outskirts of its galaxy, the light wave gets “compressed” since its front end “slows” down. By the time its back end hits the outskirts, its wavelength is shortened to

$$\begin{cases} \lambda_1 = \lambda_0 \frac{c_1}{c_0} = \frac{\lambda_0}{\sqrt{a_1}} \\ c_1 = \frac{c_0}{\sqrt{a_1}} \end{cases} \quad (356)$$

The light wave then began to expand together with the cosmos as it was on its transit toward Earth. By the time it reaches the outskirts of the Milky Way at which location the scale factor is  $a_2$  and the light speed is  $c_2 = \frac{c_0}{\sqrt{a_2}}$ , it would have undergone a cosmic expansion of  $\frac{a_2}{a_1}$ . Thus its wavelength is expanded to

$$\begin{cases} \lambda_2 = \frac{a_2}{a_1} \lambda_1 = \frac{a_2}{a_1^{3/2}} \lambda_0 \\ c_2 = \frac{c_0}{\sqrt{a_2}} \end{cases} \quad (357)$$

As the photon enters the Milky Way in which the light speed remains  $c_0$  (because the Milky Way has resisted expansion), the front end of the light wave races toward Earth, thus “stretching” out the wavelength to

$$\lambda_3 = \lambda_2 \frac{c_0}{c_2} = \frac{a_2}{a_1^{3/2}} \lambda_0 \sqrt{a_2}, \quad (358)$$

or

$$\frac{\lambda_3}{\lambda_0} = \left( \frac{a_2}{a_1} \right)^{3/2}. \quad (359)$$

The light wave, having been “stretched” out, can no longer be absorbed by the respective atom on Earth. The light wave is said to have experienced a redshift. The redshift defined as  $z \triangleq \frac{\lambda_{\text{observed}} - \lambda_{\text{emit}}}{\lambda_{\text{emit}}} = \frac{\lambda_3 - \lambda_0}{\lambda_0}$  thus satisfies

$$1 + z = \left( \frac{a_2}{a_1} \right)^{3/2}. \quad (360)$$

What is new in this context, as compared with standard cosmology, is the exponent of  $\frac{3}{2}$  in the redshift formula instead of unity. Because  $a_2 > a_1$ ,  $\left( \frac{a_2}{a_1} \right)^{3/2} > \frac{a_2}{a_1}$ , meaning that the redshift as detected on Earth gets enhanced as compared with the raw ratio of the scale factors  $\frac{a_2}{a_1}$ . To deduce the actual ratio  $\frac{a_2}{a_1}$ , we must reduce the detected redshift value

accordingly. Without this knowledge, one would mistakenly overestimate the redshift as has been an issue in the Hubble law and the calibration of the Hubble constant.

Note that we allow the expansion of the outer space to freely take place even before the galaxies were formed (i.e., we did not presume  $a_1 = 1$ ). Only the ratio  $\frac{a_2}{a_1}$  between the cosmic scale factors in the outer space appears in the redshift formula, however. We also do not make any specific assumptions about the actual cosmic expansion process during the transit of the light wave. <sup>40</sup>

## References

- [1] P. Hořava, Phys. Rev. D 79, 084008 (2009)
- [2] P.D. Mannheim and D. Kazanas, Astro. J 342, 635-638 (1989)
- [3] P.D. Mannheim, Prog. Part. Nucl. Phys. 56, 340 (2006)
- [4] P.D. Mannheim and J.G. O'Brien, Phys. Rev. Lett. 106, 121101 (2011)
- [5] P.D. Mannheim and J.G. O'Brien, *Fitting galactic rotation curves with conformal gravity and a global quadratic potential*, arXiv:1011.3495v2
- [6] J.G. O'Brien and P.D. Mannheim, *Fitting dwarf galaxy rotation curves with conformal gravity*, arXiv:1107.5229
- [7] K. Schwarzschild, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften 1, 189 (1916)
- [8] P.D. Mannheim, Foundations of Physics 42, 388 (2012)
- [9] B. Fiedler and R. Schimming, Rept. Math. Phys. 17, 15 (1980)
- [10] H.A. Buchdahl, Il Nuovo Cimento, Vol. 23, No 1, 141 (1962)
- [11] H.A. Buchdahl, Mon. Not. R. astr. Soc. 150, 1 (1970)
- [12] D.R. Noakes, J. Math. Phys. 24, 1846 (1983)
- [13] P. Teyssandier and Ph. Tournenc, J. Math. Phys. 24, 2793 (1983)
- [14] T.P. Sotiriou and V. Faraoni, *f(R) theories of gravity*, Rev. Mod. Phys. 82, 451 (2010)
- [15] K.K. Nandi and A. Bhadra, Phys. Rev. Letts. 109, 079001 (2012)
- [16] A. Bhattacharya, G.M. Garipova, E. Laserra, A. Bhadra, and K.K. Nandi, JCAP 1102:028 (2011)
- [17] R. Isaev, A.A. Potapov, and K.K. Nandi, *Fixing a Parameter of the Galactic Halo: A Mathematical Modeling by Hamiltonian Method*, arXiv:1108.0926
- [18] A. Bhattacharya, R. Isaev, K.B. Vijayakumar, and K.K. Nandi, *Upper Limit on the Size of Galactic Halo via Hamiltonian Approach*, arXiv:1003.0165
- [19] J. Sultana and D. Kazanas, Phys. Rev. D 81, 127502 (2010)
- [20] J. Sultana, D. Kazanas, and J.L. Said, Phys. Rev. D 86, 084008 (2012)
- [21] J.L. Said, J. Sultana, and K.Z. Adami, Phys. Rev. D 85, 104054 (2012)
- [22] Y. Brihaye and Y. Verbin, Phys. Rev. D 81, 124022 (2010)
- [23] R.K. Nesbet, *Conformal gravity: dark matter and dark energy*, arXiv:1208.4972
- [24] S. Pireaux, Class. Quant. Grav. 21, 1897 (2004)
- [25] E.E. Flanagan, Phys. Rev. D 74, 023002 (2006)

<sup>40</sup> Our final comment is on the galactic gravitational resistance to cosmic expansion. It is agreed that galaxies resist the cosmic expansion; otherwise, the redshift of light from distant galaxies would not be detectable. Yet the mechanism for the gravitational confinement is not well addressed in standard cosmology. It is often a hand-waving notion that the presence of matter holds the “fabric” of space inside a galaxy stationary, thus enabling the galaxy to resist the cosmic expansion. What constitutes the “fabric” is often left unanswered. Curvature-scaling gravity answers this question. The “fabric” is nothing but the Ricci scalar. It is reasonable to expect that due to the presence of matter,  $\mathcal{R}$  within galaxies is higher than that in the void of the outer space, and that with galaxies stabilizing due to their own rotations,  $\mathcal{R}$  inside galaxies tends to be stationary as well.

- [26] S. Capozziello, M. De Laurentis, and V. Faraoni, *A bird's eye view of  $f(\mathcal{R})$ -gravity*, arXiv:0909.4672
- [27] S. Capozziello and M. De Laurentis, *Extended Theories of Gravity*, Phys. Rept. 509, 167 (2011)
- [28] R. Schimming and H.-J. Schmidt, *On the history of fourth order metric theories of gravitation*, arXiv:gr-qc/0412038
- [29] H.-J. Schmidt and D. Singleton, *Isotropic universe with almost scale-invariant fourth-order gravity*, arXiv:1212.1769
- [30] F.S.N. Lobo and T. Harko, *Extended  $f(\mathcal{R}, \mathcal{L}_m)$  theories of gravity*, arXiv:1211.0426
- [31] O. Bertolami and J. Páramos, Class. Quant. Grav. 25, 245017 (2008)
- [32] S. Capozziello, N. Frusciante, and D. Vernieri, *New spherically symmetric solutions in  $f(\mathcal{R})$ -gravity by Nöther symmetries*, arXiv:1204.4650
- [33] T. Multamaki and I. Vilja, Phys. Rev. D 74, 064022 (2006)
- [34] K.S. Stelle, Phys. Rev. D 16, 953 (1977)
- [35] G.D. Birkhoff, *Relativity and Modern Physics*, Cambridge, MA: Harvard University Press (1923)
- [36] T. Clifton, P.G. Ferreira, A. Padilla, and C. Skordis, Physics Reports 513, 1 (2012)
- [37] T. Clifton, Class. Quant. Grav. 23 (2006) 7445
- [38] G. Varieschi, Phys. Res. Int. 2012:469095 (2012)
- [39] G. Varieschi, ISRN Astronomy and Astrophysics, vol. 2011, 806549 (2011)
- [40] G. Varieschi, Gen. Relativ. Gravit. 42, 929 (2010)
- [41] V. Perlick and C. Xu, Astro J 449, 47 (1995)
- [42] G.F.R. Ellis, Gen. Rel. Grav. 39, 511-520 (2007)
- [43] A. Friedmann, Z. Phys. 10, 377 (1922)
- [44] A. Friedmann, Z. Phys. 21, 326 (1924). See also [50] or [51]
- [45] E. Hubble, Proceedings of the National Academy of Sciences of the United States of America, Vol 15, March 15, 1929: Issue 3, 168. See also [50] or [51]
- [46] H.P. Robertson, Astrophysical Journal 82: 284 (1935)
- [47] H.P. Roberson, Astrophysical Journal 83: 187 (1936)
- [48] H.P. Robertson, Astrophysical Journal 83: 257 (1936)
- [49] A.G. Walker, Proceedings of the London Mathematical Society 2 42 (1): 90 (1937). See also [50] or [51]
- [50] S. Weinberg, *Gravitation and cosmology: principles and applications of the general theory of relativity*, John Wiley & Sons Inc. (1972)
- [51] D.S. Gorbunov and V.A. Rubakov, *Introduction to the theory of the early universe: Hot Big Bang theory*, World Scientific (2011)
- [52] J. W. Moffat, Int. J. Mod. Phys. D 2, 351 (1993); arXiv:gr-qc/9211020
- [53] A. Albrecht and J. Magueijo, Phys. Rev. D 59, 043516 (1999)
- [54] J. Magueijo and L. Smolin, Phys. Rev. Lett. 88, 190403 (2002)
- [55] J. Magueijo and J. W. Moffat, Gen. Rel. Grav. 40, 1797-1806 (2008)
- [56] A. Guth, Phys. Rev. D 23, 347 (1981)
- [57] P.J. Steinhardt, *The inflation debate*, Scientific American 304, 36 (2011)
- [58] A. Riess et al, Astron. J. 116, 1009 (1998)
- [59] S. Perlmutter et al, ApJ 517, 565 (1999)

- [60] A. Riess et al, arXiv:astro-ph/0611572
- [61] M.S. Turner and A. Riess, *Astrophys. J.* 569, 18 (2002)
- [62] C. Rovelli, *Quantum Gravity*, Cambridge University Press (2004)
- [63] R.P. Woodard, *Lect. Notes Phys.* 720, 403 (2007)
- [64] S. Capozziello and S. Vignolo, *Class. Quant. Grav.* 26, 175013 (2009)
- [65] A.D. Dolgov and M. Kawasaki, *Phys. Lett. B* 573, 1 (2003)
- [66] D. Comelli, *Phys. Rev. D* 72, 064018 (2005)
- [67] I. Navarro and K. Van Acoleyen, *JCAP* 0603, 008 (2006)
- [68] C.M. Bender and P.D. Mannheim, *Phys. Rev. D* 84, 105038 (2011)
- [69] H. Ohanian, *Einstein's mistakes: the human failings of genius*, p154, W. W. Norton (2008)
- [70] M. Milgrom, *Ap. J.* 270, 365, 371, 384 (1983)
- [71] J.W. Moffat, *J. Cos. Ast. Phys.* 5, 3 (2005)